

University of Bahrain

Department of Physics

**PHYCS425: Computational Physics**

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**Constrained motion on surfaces**

**Instructor**

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1. **Introduction**

Imagine you’re on a roller coaster. You’re plunging downwards from a ramp at dizzying speeds. You’re terrified, but not afraid for your life[[1]](#footnote-1). You know that no matter what happens, you’ll always remain on the track. This is an example of constrained motion, and it is virtually everywhere. An athlete skiing down an icy mountain, a car driving down a road[[2]](#footnote-2) are instances of constrained motion on surfaces.

There are other forms of constrained motion, like that of a fluid moving in a pipe. But our focus in this project is only on the former. These types of constraints fall under the category of “holonomic constraints”. For each one of these constraints, the system loses a degree of freedom.

If one wants to investigate such motion analytically, Newton’s laws are not really helpful. One usually resorts to Lagrangian or Hamiltonian mechanics to get any sort of insight from these systems. Even then, getting exact solutions is cumbersome, and when damping is added, it becomes exponentially more difficult to obtain. This is where computer simulations come in handy. They allow us to study the properties of motion without knowing the most general form of the solution.

In this project, we will explore constrained motion on various surfaces in both 1D & 2D. In part I, we’ll investigate the motion on 2 curves: a parabola (), and an unknown curve , which we have to identify. We also identify the equilibrium points for both. We study the periodic motion in detail, exploring the effect of various parameters like the dimension of the curves & initial displacement. We then add damping and driving forces to the systems and investigate the possibility of chaotic motion.

In part II, we have a spherical surface. We investigate the initial conditions that lead various paths – including closed paths and Lissajous-like figures. Here too, we investigate if chaotic motion is possible. In addition, we study the effects of a homogeneous damping force.

In all parts, we used 4th order RK4 method to simulate the systems. The conservation of energy and angular momentum were checked wherever relevant.

Buckle up, because you don’t want to fall off this roller coaster!

1. **Theory**

**2.1- The Runge-Kutta Method**

Since we used 4th order Runge-Kutta method (or RK4) throughout this project, it is worthwhile to go over some aspects of this method. But first, some historical context. Back in 1895, Carl Runge thought he could outsmart Euler by coming up with a numerical method more accurate than Euler. Euler’s method was to approximate solutions to initial value problems by computing the slopes of the curve at the starting point, and so the tangent line. This how Euler’s method work. What did Runge come up with to approximate the solutions to initial value problems by adding the weighted average of four increments (only one increment in Euler’s method). Those increments (denoted by ki) are the product of the step size times the estimated slope specified by the differential equation. The first increment is the same as Euler’s one. Second and third increments are the slopes at the midpoint of the step size, followed by the slope at the end of the interval.

[ADD THE EQUATIONS OF K’S FROM WIKIPEDIA]

The RK4 is the fourth-order method, means that the local truncation error is in the order of *O(h5)*, while the total accumulated error is in the order *O(h4)*.

If the differential equation we’re approximating doesn’t have explicitly the independent variable, then the system called “as physicists” time-invariant system.

**2.2- Holonomic constraints**

Constrained motion in 1D or 2D surfaces are examples of holonomic systems. Holonomic system is a system which you can deduce the state of system by only knowing the information about the change of positions of the system’s components over time, without needing to know the velocities they are moving relative to each other. To define a holonomic constraint, it is the relation between position variables that expressed as:

[ADD THIS EQUATION: *f(x,y,z)=0*]

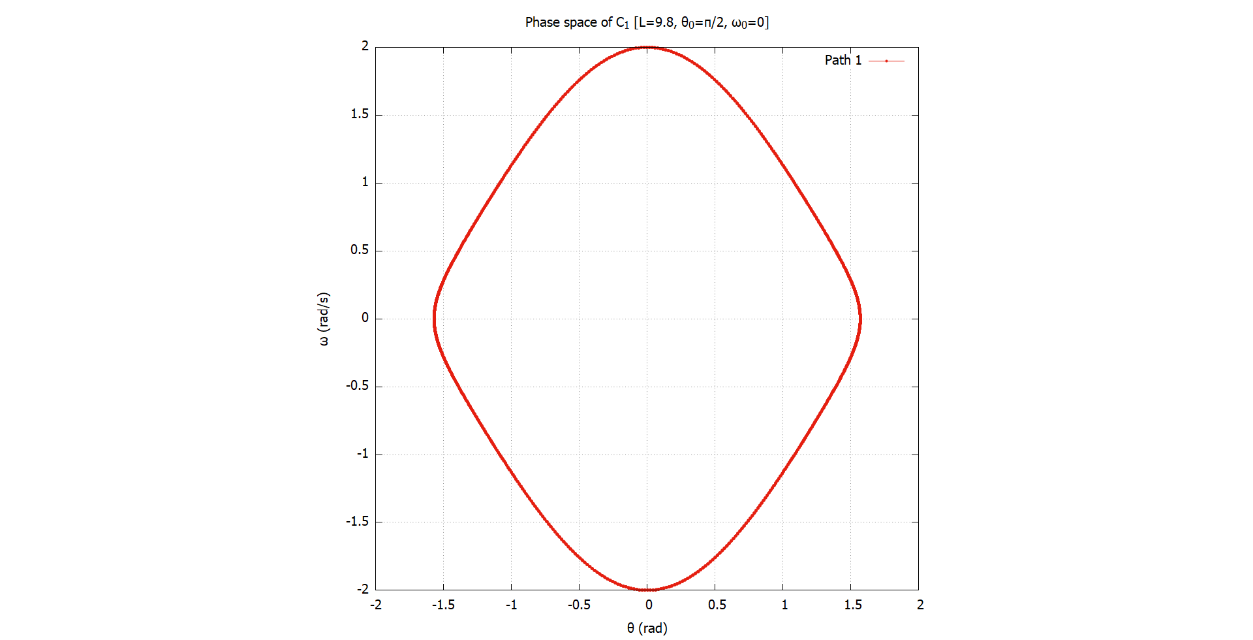
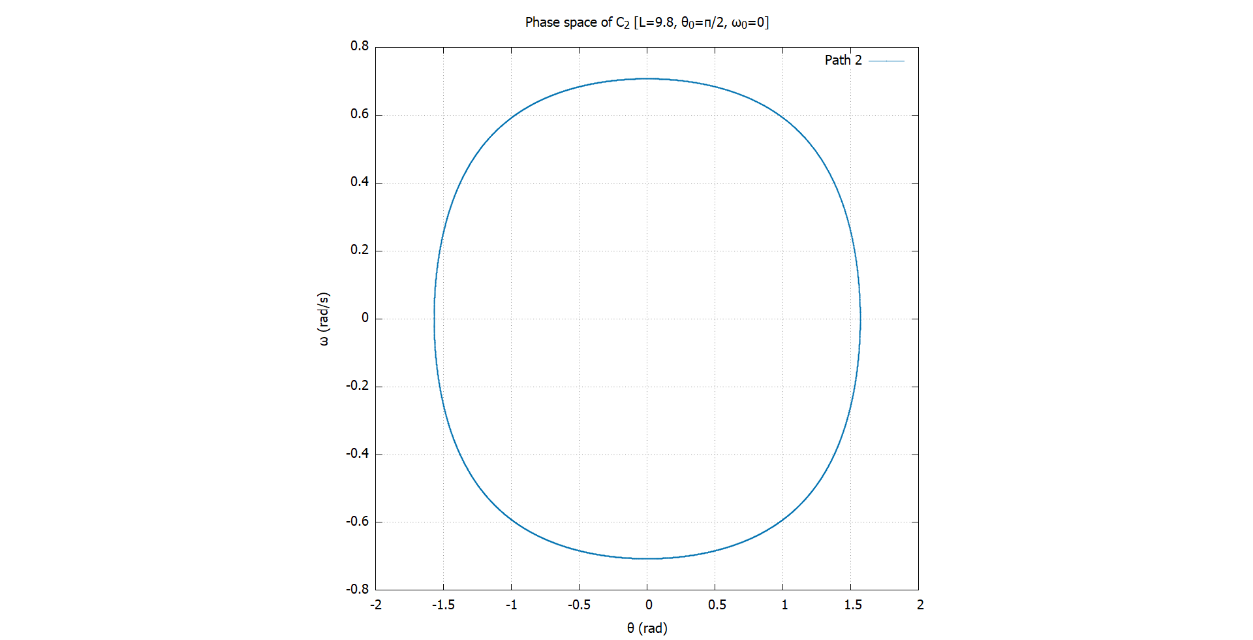
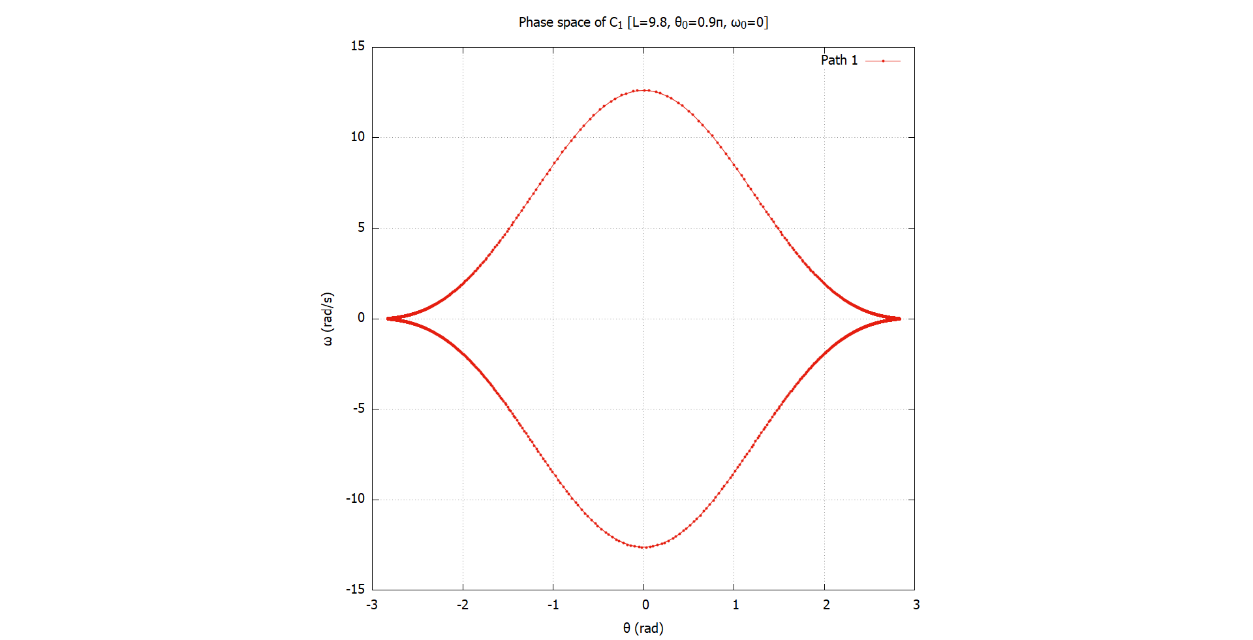
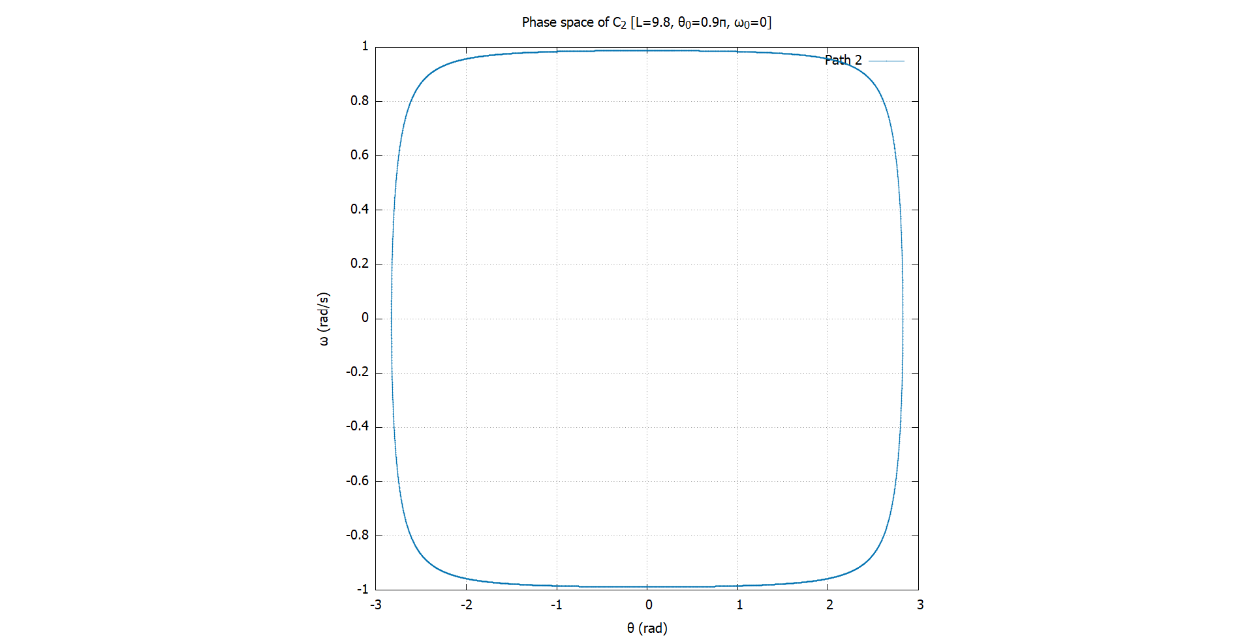
Where x, y, and z are position components in cartesian coordinates (it could be by any coordinate system). And if the system obeys a holonomic constraint, it is holonomic system.

Consider a simple pendulum, which is a system composed of weight and string. Being the length of the string fixed, then the system is holonomic since it obeys the constraint:

[x2+y2 – L2=0] [ADD THE PICTURE OF SIMPLE PENDULUM FROM WIKIPEDIA PAGE OF HOLONOMIC CONSTRAINT]

1. **Part I: Constrained motion in 1D**

**3.1- Oscillatory motion & equilibrium points**

 Trying out different initial conditions, we could confirm that a particle undergoes oscillatory motion on both curves . This corresponds to closed paths in the phase space. Shown below are the phase space portraits corresponding to the different initial conditions for both curves.

Closed paths in the phase space correspond to oscillatory motion. For , this is replicated for all initial conditions in the allowed range .[[3]](#footnote-3)

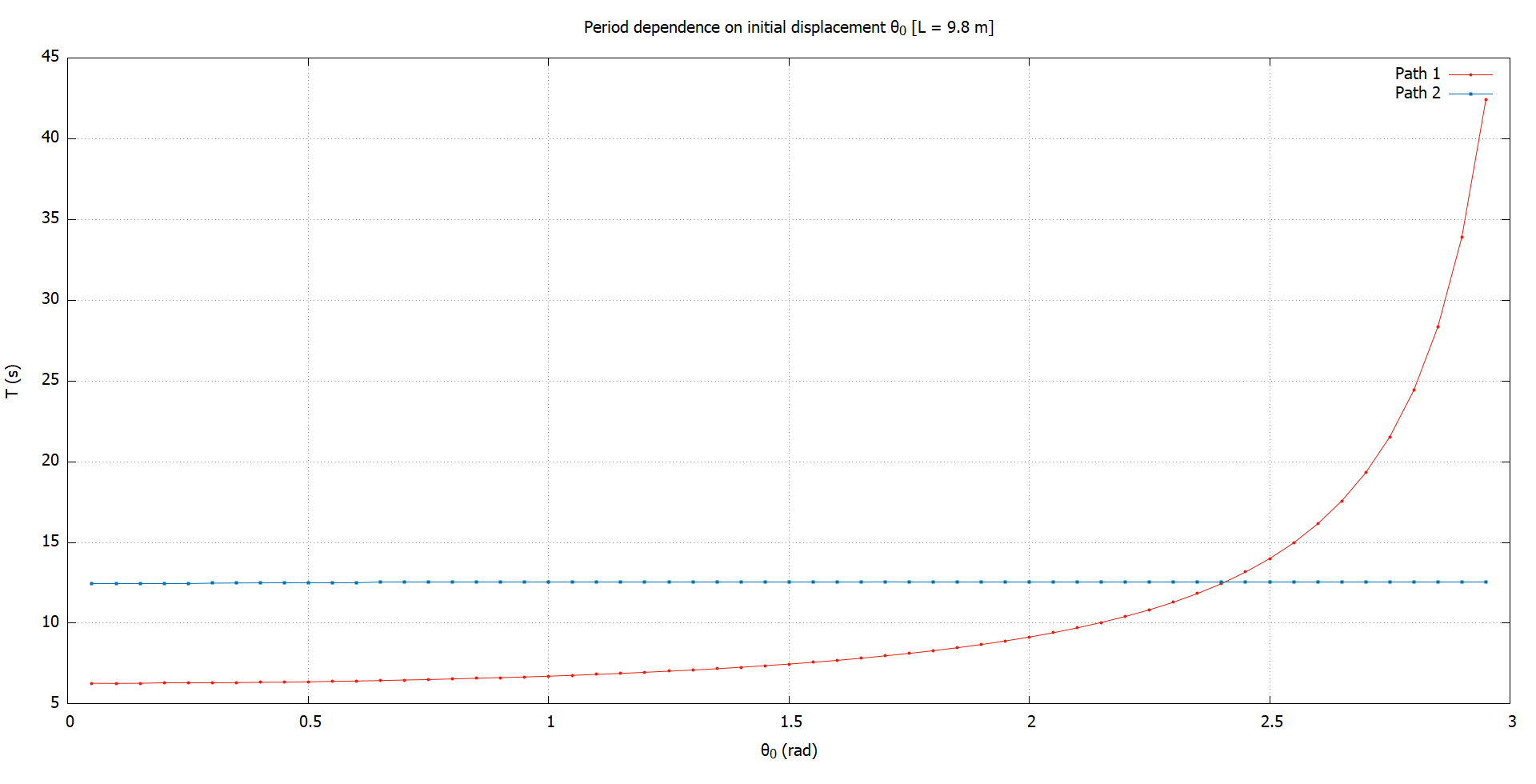
For , if the initial velocity exceeds a certain limit, the motion is no longer constrained, and the object escapes the curve entirely and it is not oscillatory.

Classically, equilibrium points correspond to the points where the potential energy of the system is either a local minima or maxima. This corresponds to the points where the net force/torque acting on an object is 0. For determining equilibrium points, we checked at what instances of motion the angular acceleration ( or der\_omega in the code) is close to 0, up to some tolerance.

We found that both curves have one equilibrium point each, corresponding to . For the parabola (), this corresponds to its vertex, the point at which the gravitational potential is minimum. And as we will find out later on in this report, the equilibrium point for is also where gravitational potential is minimum.

Knowing that the potential energy at equilibrium points is minimum, we can conclude that these points are points of *stable equilibrium*. This will be further justified when we look at the phase portraits of damped motion.

**3.2- Dependence of oscillation period () on the oscillation amplitude ()**

 For , we find that the period increases steadily as the oscillation amplitude increases. Remarkably, for , the period remains constant as we change the oscillation amplitude. This is an important clue that will be used to identify later on. The graph below shows the change in period when

To get this graph, we had to figure out a way to measure the period from the code. We summarise our method below.

Firstly, we note that is symmetric about . This is also evidenced in the phase portraits above. We can therefore study the effect of amplitude between . The behaviour of period will be mimicked in . This symmetry also applies to , so we can use this for as well.

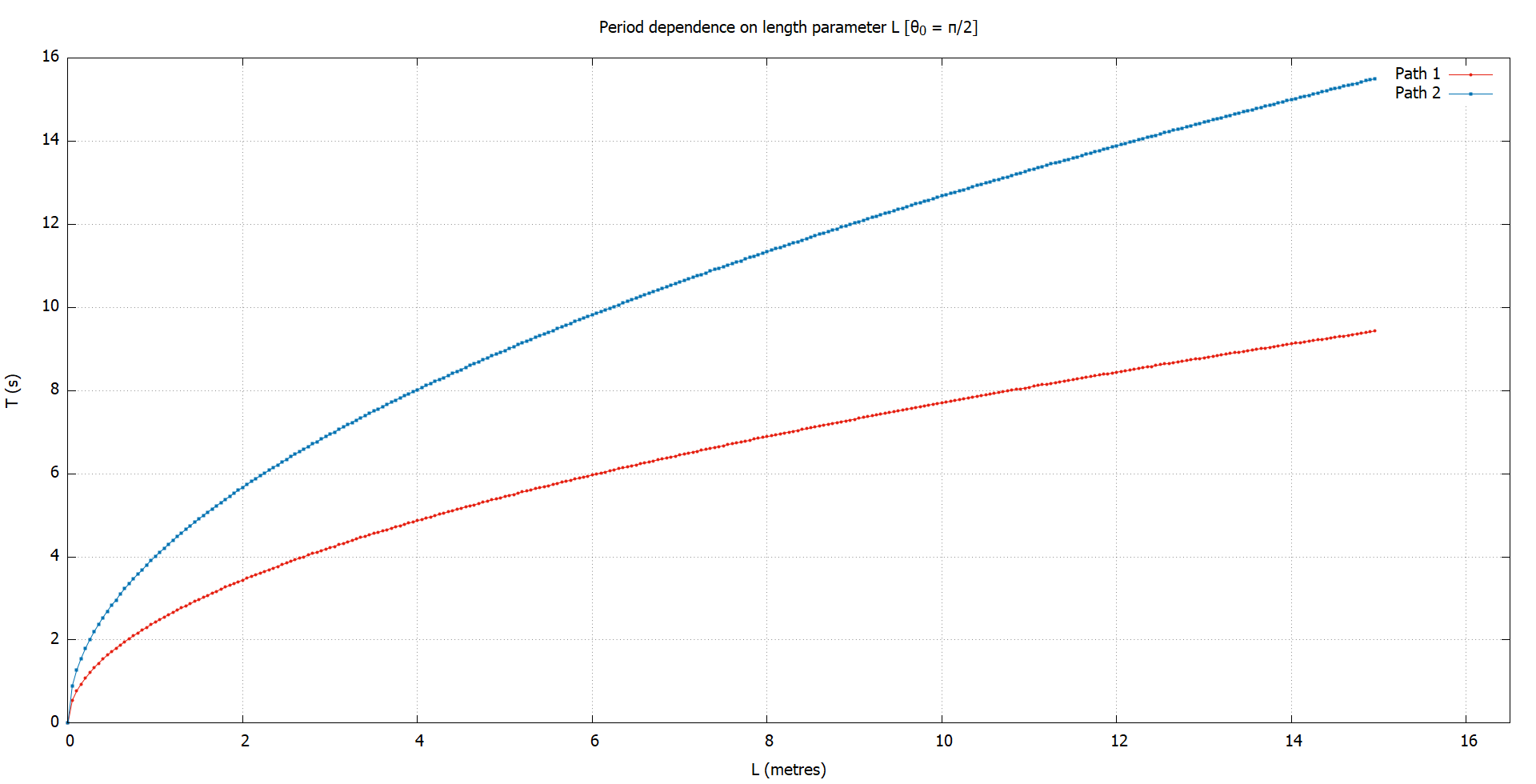
To simplify the period measurement, we choose the initial angular velocity to be 0. This means that the initial displacement corresponds to the oscillation amplitude. To measure the period (), we checked when the system returned to its initial conditions (). This meant checking when angular velocity came close to 0 (), up to some tolerance and .

The result isn’t always reliable, as sometimes we would get multiples of instead of itself. This makes sense, as the system returns to its initial condition every interval. But it made the analysis more difficult (but still worth it).

**Fitting the graph for :**

* For , the relation is quite straightforward. The period is constant around . So, we may write:
* For , the relation is much less obvious. For all we know, there may not even be a closed form expression for the period. After fiddling around randomly with various functions, we struck (fool’s) gold. When we try to fit the graph above with the equation below, we get a really good fit (~3.36% error).

**3.3- Dependence of oscillation period () on the length parameter ()**

We repeat the same analysis as above for this part. Except here, we loop over different length parameters instead of oscillation amplitude. The period measurement is done in e exact same way. This means that we were plagued by the same problems as before (measuring instead of ). This is the graph we obtained for

The results here are a bit more similar. The period increases as we increase . Physically, this makes sense, as making the curve bigger means a longer distance to travel, increasing the period. We find that for both , the period is proportional to the square root of . This was verified by fitting the data using expressions of the form in Gnuplot.

**3.4- (Optional) General expression for period () as a function of**

* **Curve :**

The equation for curve is quite straight forward. From section 3.3, we saw that . With a bit of dimensional analysis, we can figure out that for the units to match on both sides, we must somehow get rid of length units and add some time units to the LHS. There is really only one parameter in our system that can do this job, and that is the acceleration due to gravity () which has units of .

We can show that:

But, from section 3.2, we know that when , . So, we may write:

With some testing, this turns out to be a really accurate expression. We suspect that this expression can be derived analytically as well, but we’ll leave that to others.

* **Curve :**

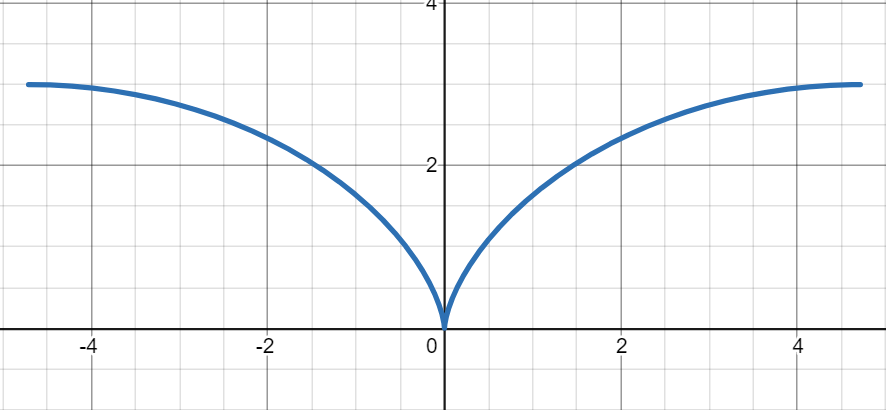
Taking inspiration from our analysis for , we may deduce a similar expression for the period of curve . We propose the following formula:

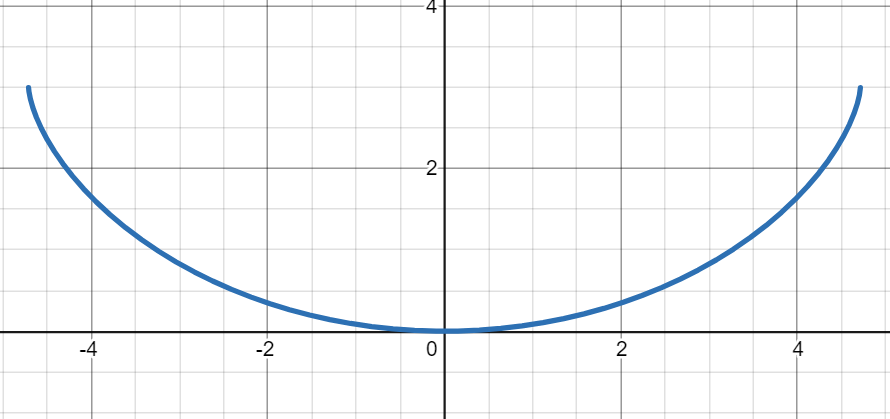
While this is by no means a great approximation, it is better than having to stab at the dark. But if someone’s life depends on the accurate prediction of period of an object moving along a parabolic ramp, don’t count on this formula.

Notice how both expressions above are symmetric with respect to . Also note how the period of curve is exactly twice that of a simple pendulum for small angles. The period of diverges to infinity at . This makes sense, as this corresponds to the end-points of the parabola, which are at infinite distance.

**3.5- Identifying the curve**

We saw back in section 3.2 that the period of curve is independent of the oscillation amplitude . This was not some run-of-the-mill coincidence. This is a property of very special curves known collectively as “Tautochrones”. For these curves, no matter where an object starts out along the curve, the time taken to reach the bottom is the same. The most famous of which is the cycloid. (Apparently there are infinitely many tautochrones)[[4]](#footnote-4)

 Naturally, we had to try the cycloid. But which one do we try? The traditional cycloid with parametric equation looks like this when we take :

Intuitively, this does not look like a surface that can have constrained motion (due to its convexity). A more plausible cycloid may be the one with the equation , which looks like this for :

Now, how do we know if this is in fact the curve that we have at hand? There is only one way to find out, and that is to derive the equation of motion and see if it results in the same one given to us.

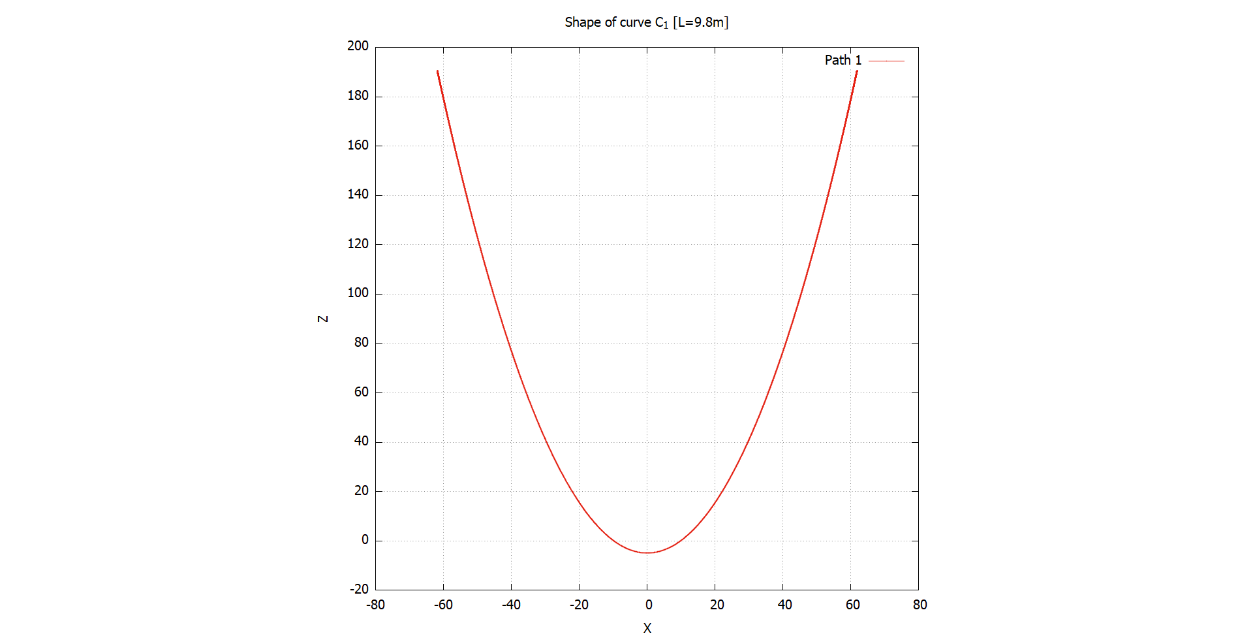
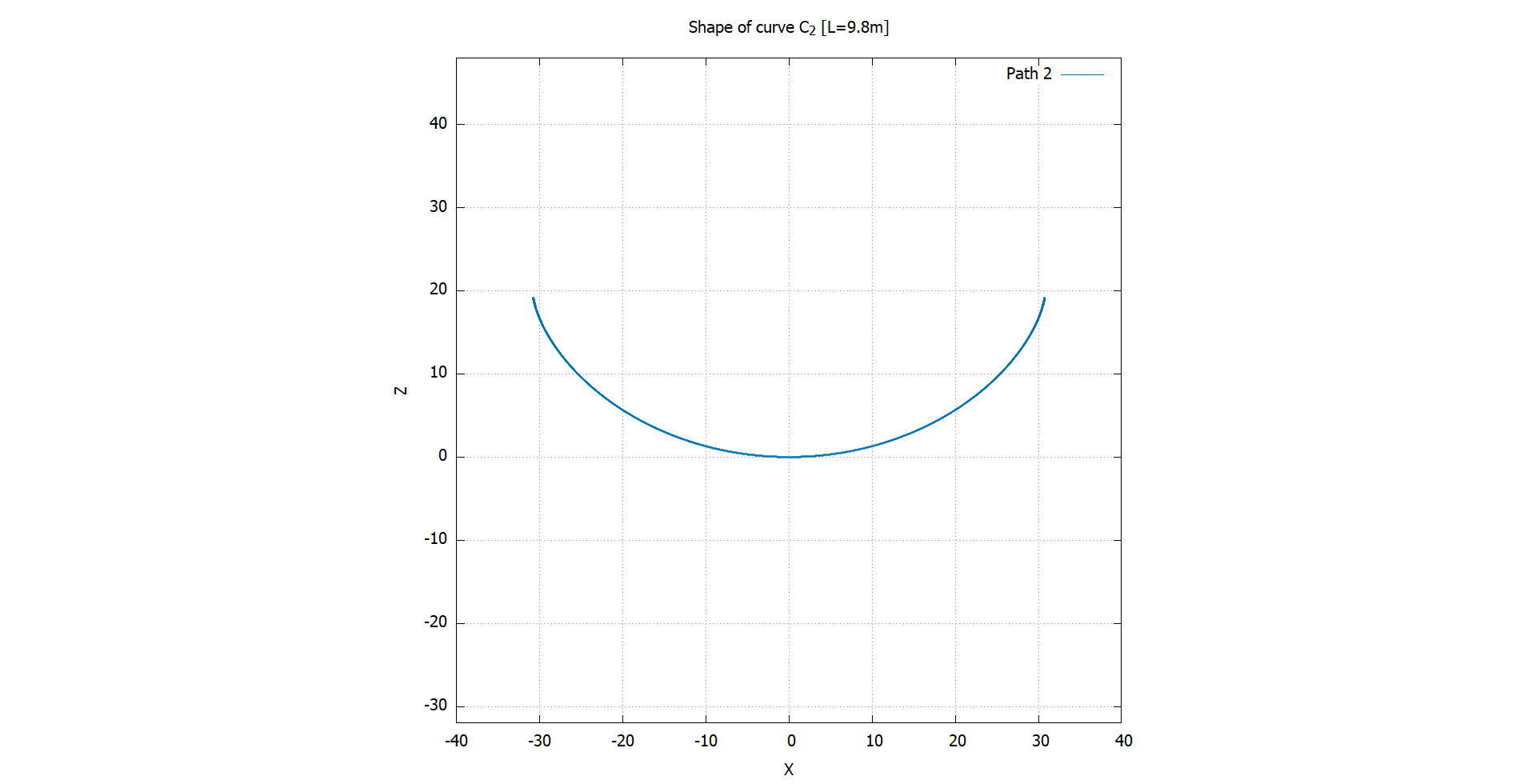
This means writing the Lagrangian of the system and obtaining the Euler-Lagrange equations for all degrees of freedom. A brief derivation is given below:

The corresponding Euler-Lagrange equation is given by (only one degree of freedom):

This is exactly identical to the equation of motion we were given, so we have our , namely: **,**

Note that translating the cardioid along by a constant doesn’t change the equation of motion. So, the cardioid in question is not unique.

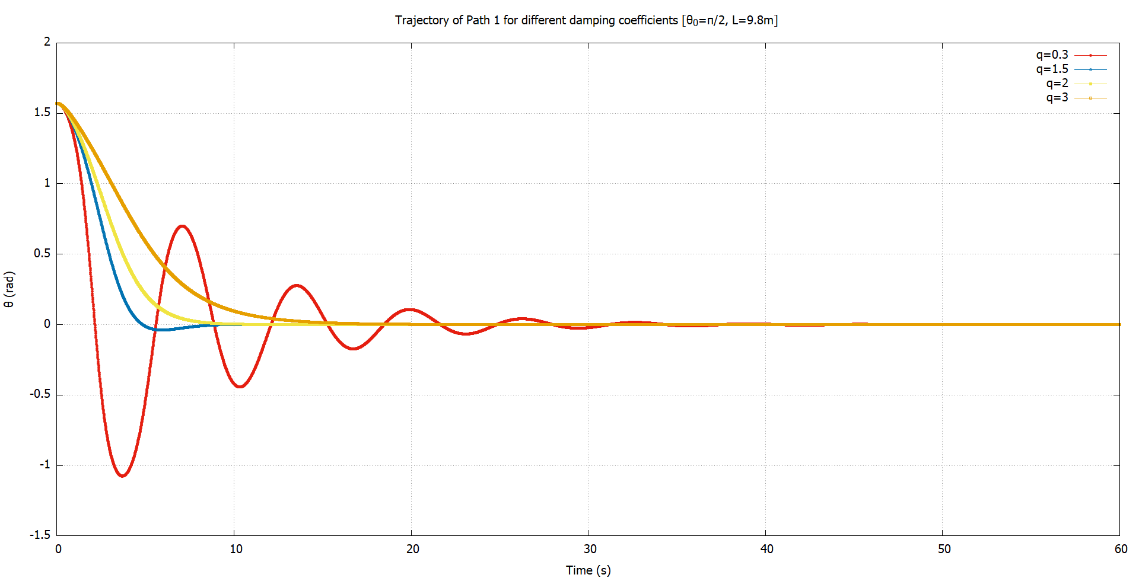
While we’re discussing the shape of the curves, let’s see what looks like. For , the origin coincides with the focus of the parabola. And as we’ve stated before, the equilibrium point () is the lowest point of the curve in each case. This corresponds to the point for and for .

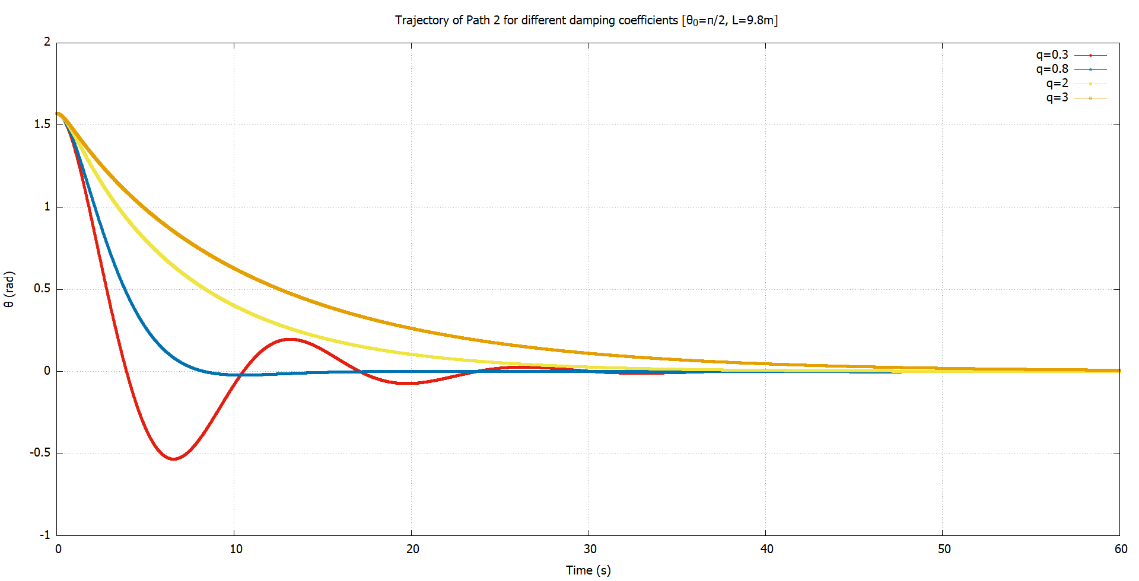


**3.6- Adding damping and driving forces, chaos (?)**

Firstly, how does one achieve a damping force for particles moving on curves practically? The solution we came up with is to imagine the entirety of the curve immersed in a viscous fluid. We assumed that the drag force is a linear one, proportional to the angular speed , but acting opposite to it. The strength of the drag force changes depending on the drag coefficient (with unit ). So, our equations of motion are now:

Like all damped systems, the systems exhibit 3 distinct behaviours: underdamped, overdamped and critically damped motion. The figures below showcase the various possibilities.[[5]](#footnote-5)

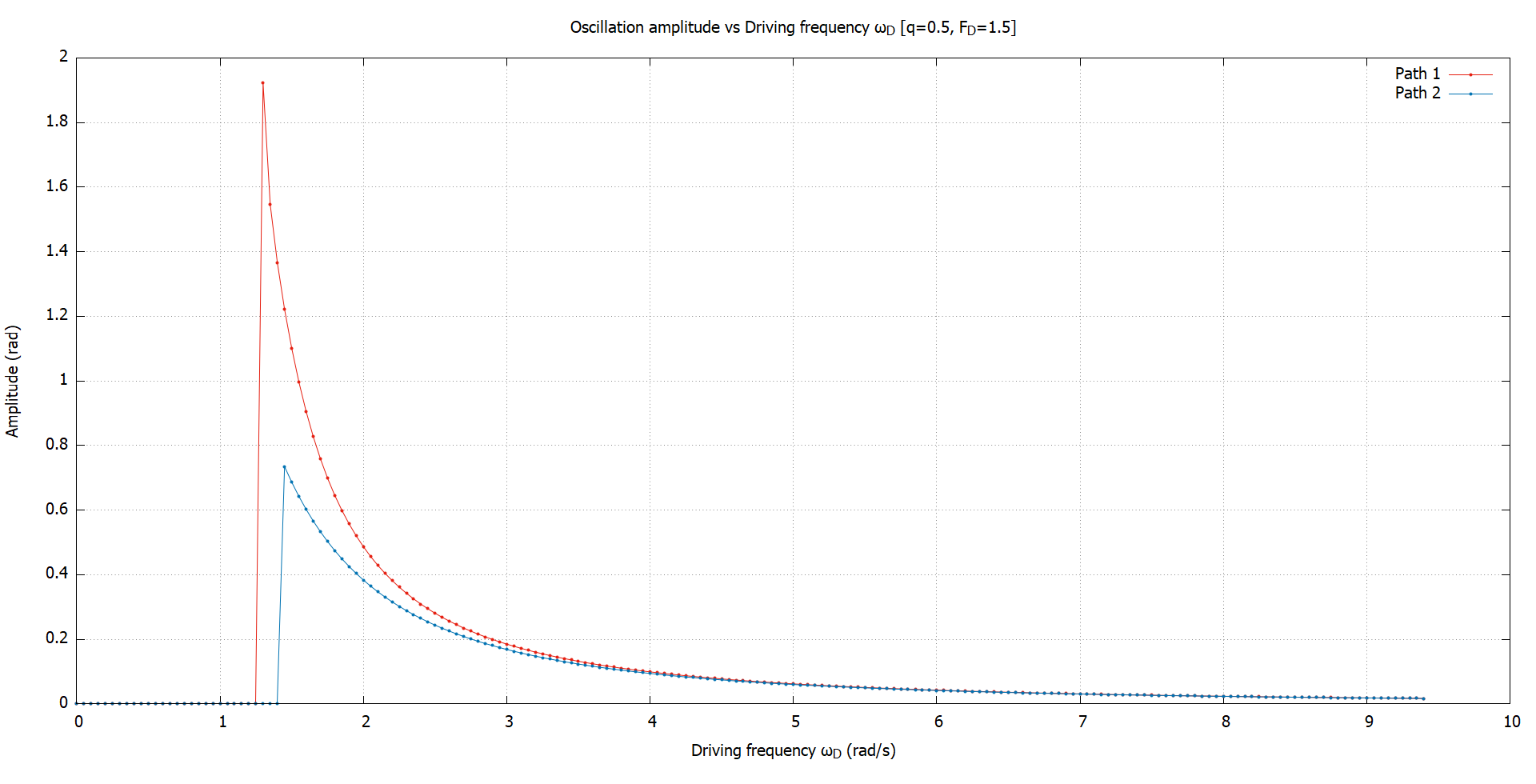




As for adding a driving force, we just acquire an additional term for both equations of motion ( has units of ). That is:

Usually, the effect of adding a sinusoidal driving force is as follows:

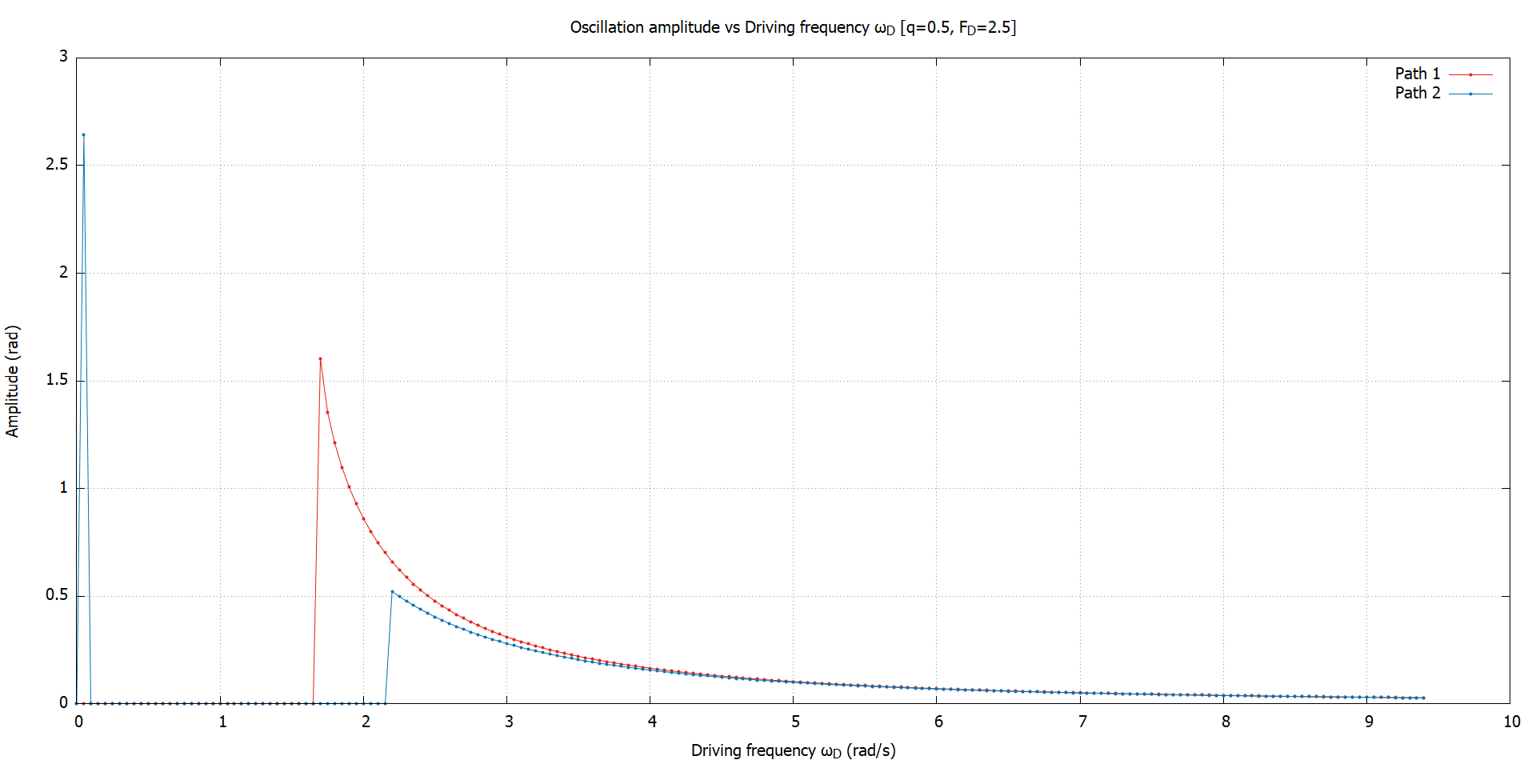
* For a fixed driving amplitude , increasing the driving frequency decreases the oscillation amplitude. This is because the same energy is shared between more oscillations, meaning less energy per oscillation.
* For a fixed driving frequency , increasing the driving amplitude increases the oscillation amplitude. Now, the system receives increasingly more energy.

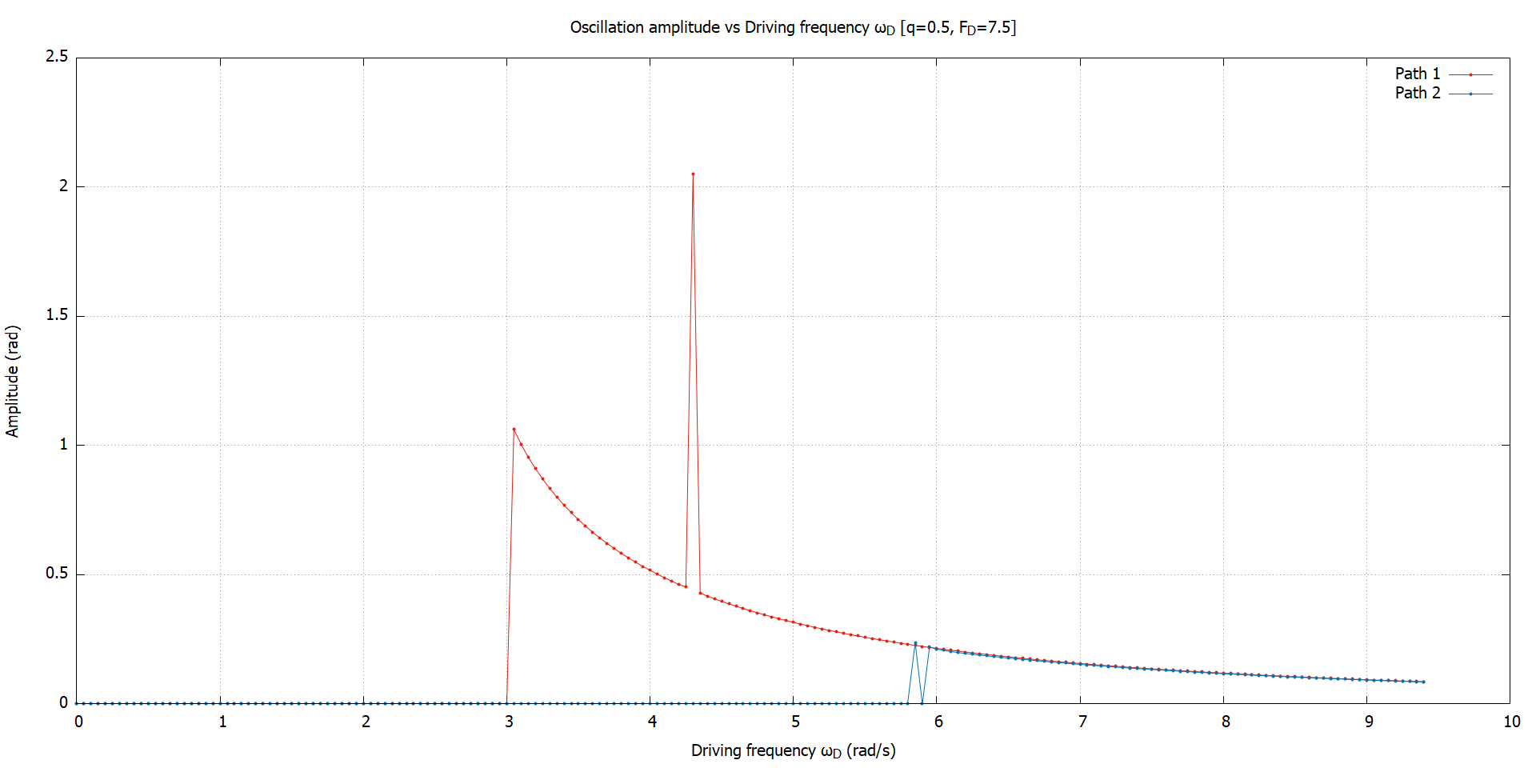
So far, so good. Let us plot the oscillation amplitude for different and see if the above holds[[6]](#footnote-6). Below we have the variation of oscillation amplitude with .

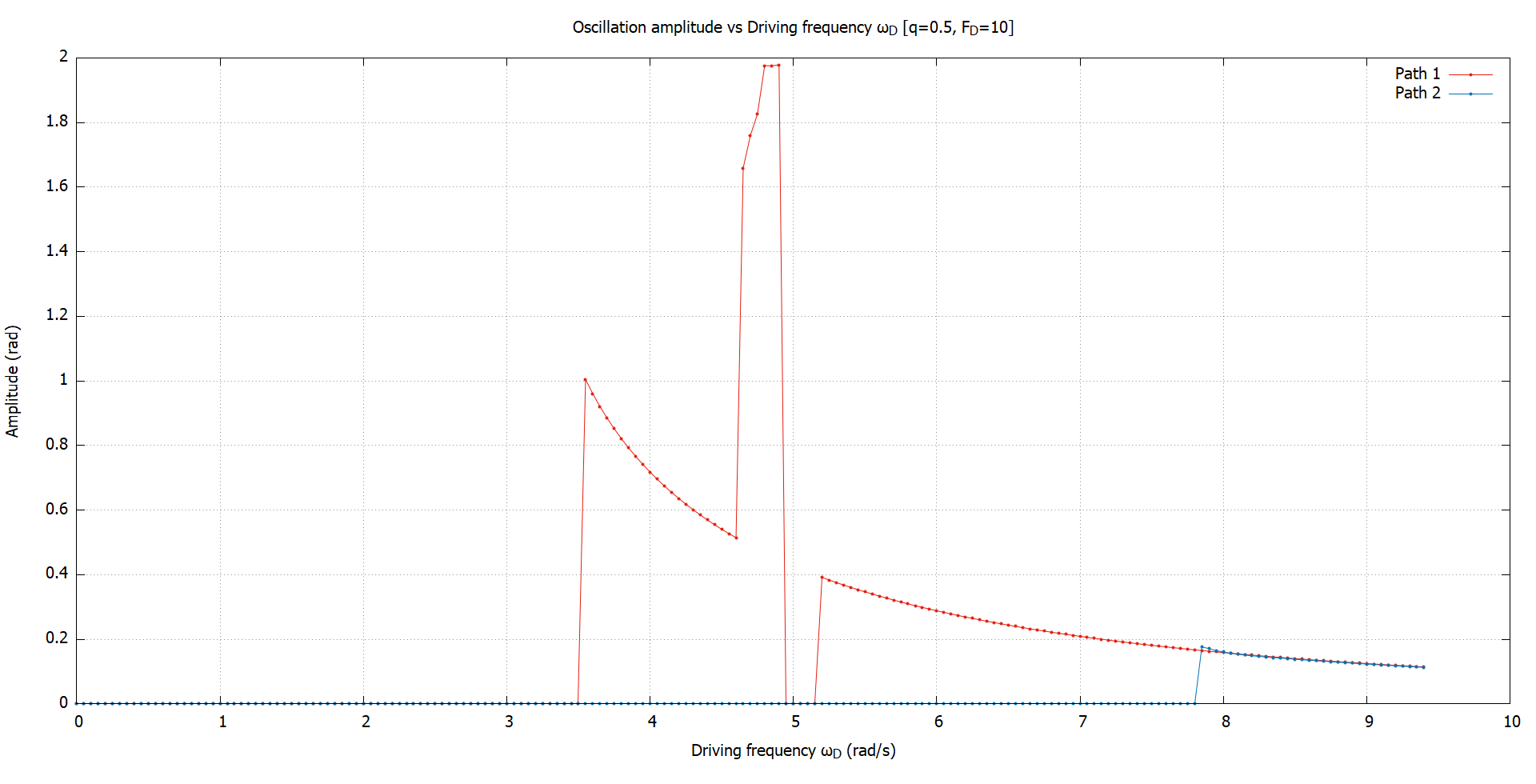
Hold on a second, I hear you ask. Why is the oscillation amplitude 0 for low frequencies? The answer is that it is not. In fact, there are no oscillations at all. When is low, the system never escapes the transient phase. shoots up to , indicating that the motion is no longer constrained. This is something that we encountered in section 3.1 for .

It makes sense that giving too much energy will cause the particle to escape for . After all, it is a curve of finite length. But why it happens for is beyond us. A parabola is a curve of infinite length. And so the particle must always stay bound to the surface, no matter what energy is given to it. Perhaps it is the inaccuracies of simulation that is to blame. In all cases, we simulated the trajectory to verify this behaviour.

Shown below are plots for oscillation amplitude vs for several other driving amplitudes .



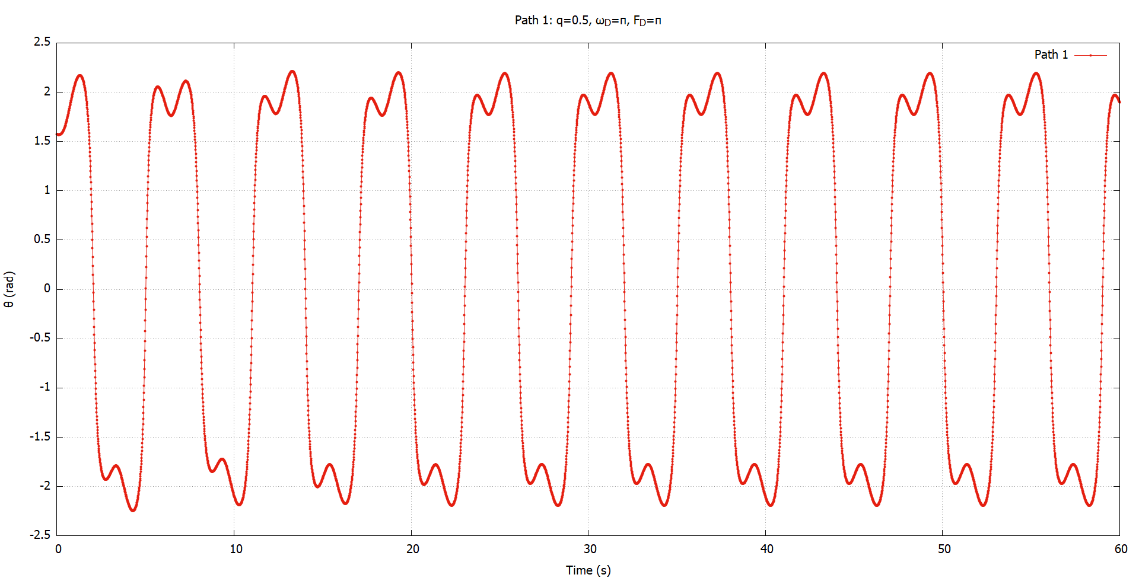


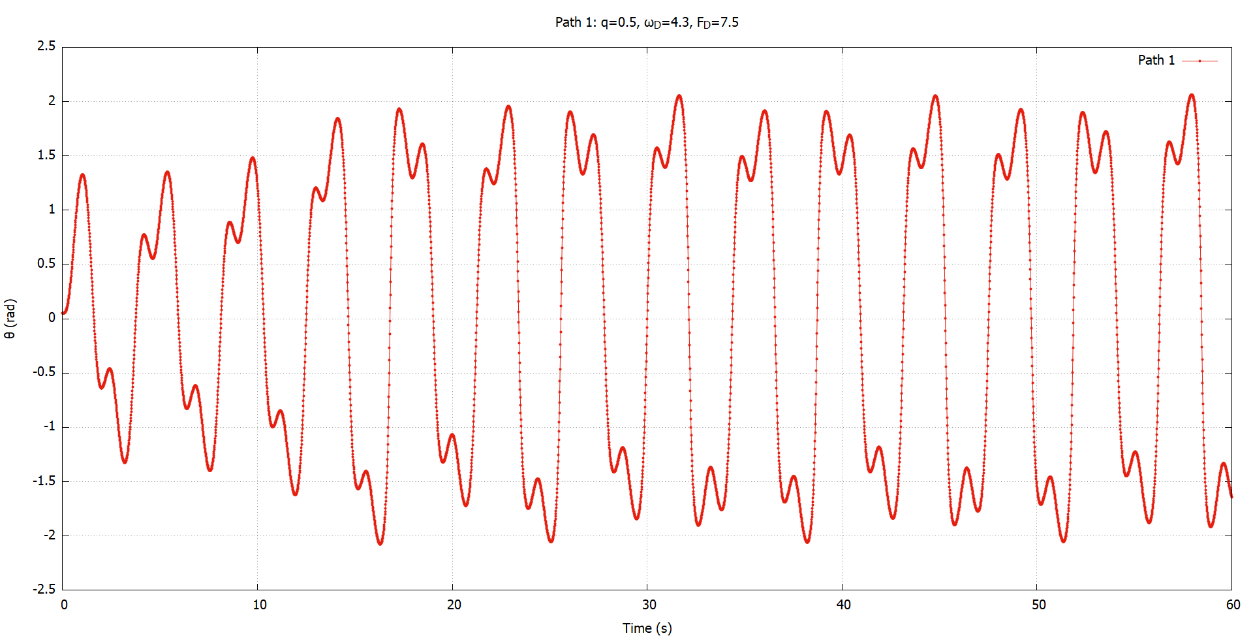
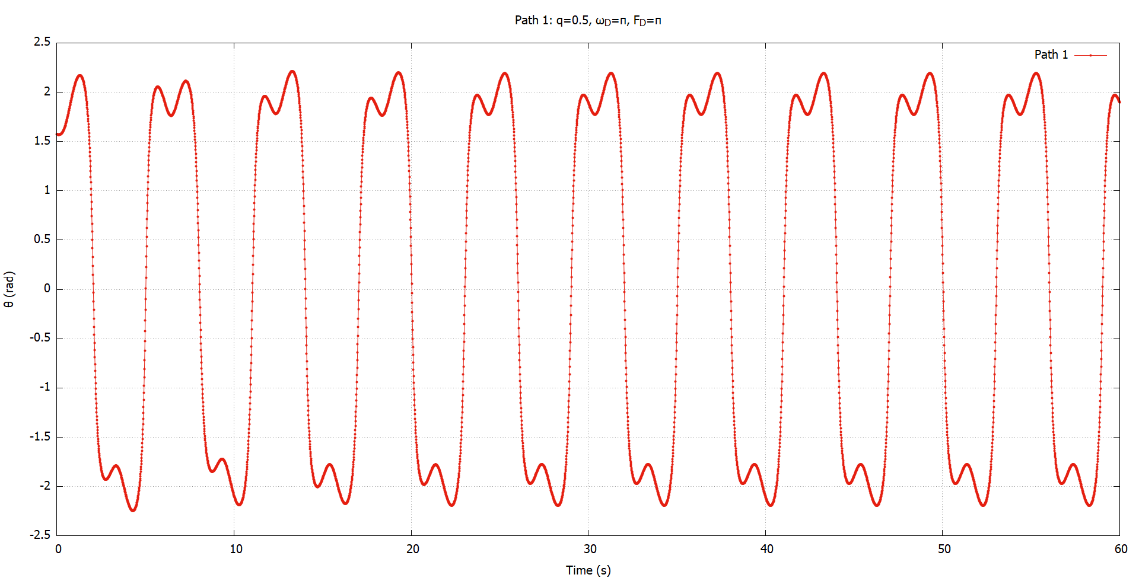


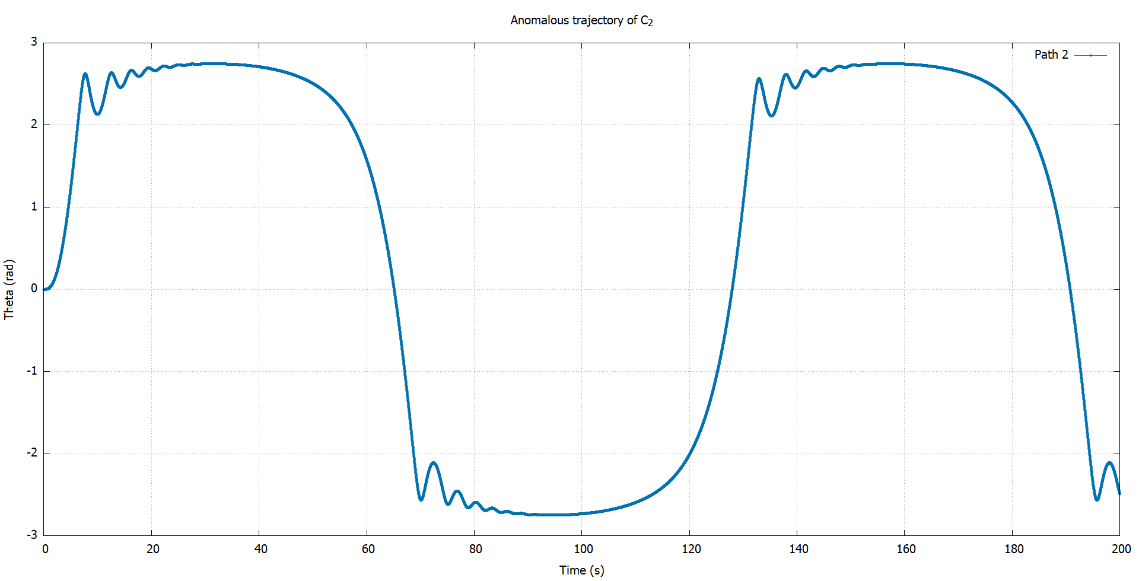
In all of the above plots, we had chosen a resolution of We felt that this was small enough to capture any unusual behaviour.

**Observations from the amplitude plots**

Firstly, if we ignore the discontinuities, there is a clear resonant frequency that is present for both curves. But those spikes are certainly annoying. The spikes occur more frequently for . These correspond, surprisingly, to “period multiplied” trajectories. While this might be an early indication of chaos, we shouldn’t get our hopes too high. The “period multiplications” don’t occur at the same . Ultimately, we hadn’t encountered any aperiodic behaviour, which is one of the markers of chaotic motion. Moreover, chaos also entails a sensitivity to initial conditions, which we also didn’t encounter.

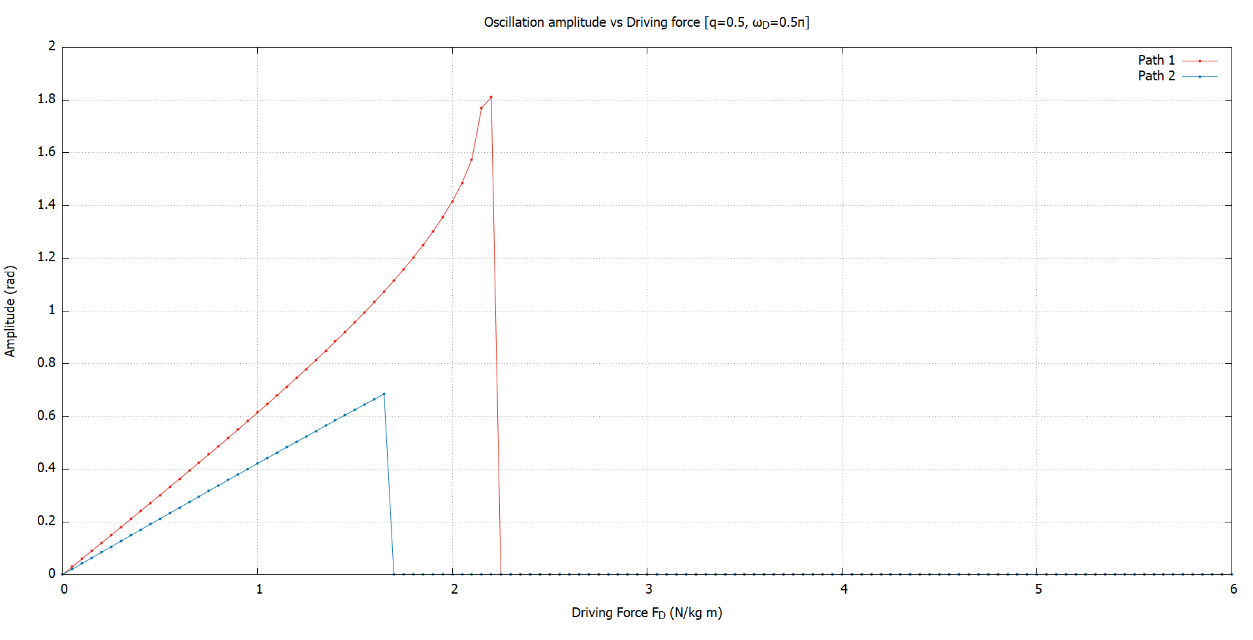
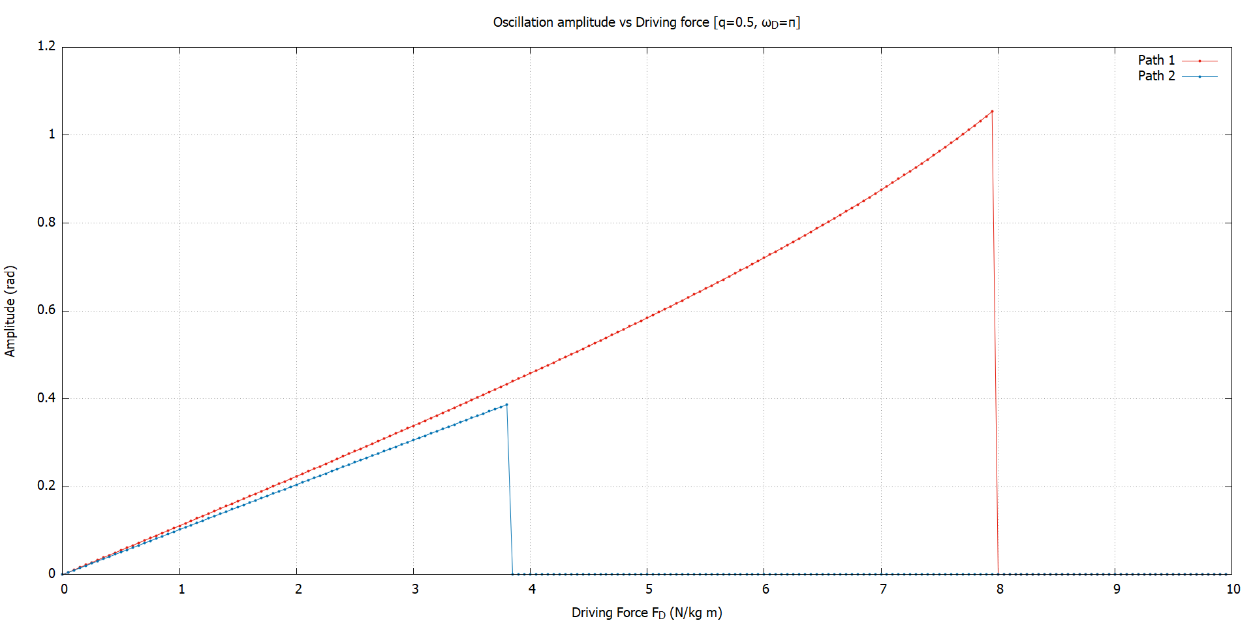
Below, we showcase some of these anomalous “period multiplied” trajectories of . Note that these trajectories are all periodic. Unlike the chaotic motion in simple pendulum, where period doubles for the same driving frequency, here period is multiplied at different .





, on the other hand, is quite well-behaved. The only anomaly we encountered is from the first graph from the previous page(). This is really on the verge of escaping the curve (). The trajectory is shown below. Note that this too, is periodic.

**Variation of amplitude with driving amplitude**

For good measure, we also plotted the oscillation amplitude variation with for fixed . These are the resulting plots. Again, the resolution was .

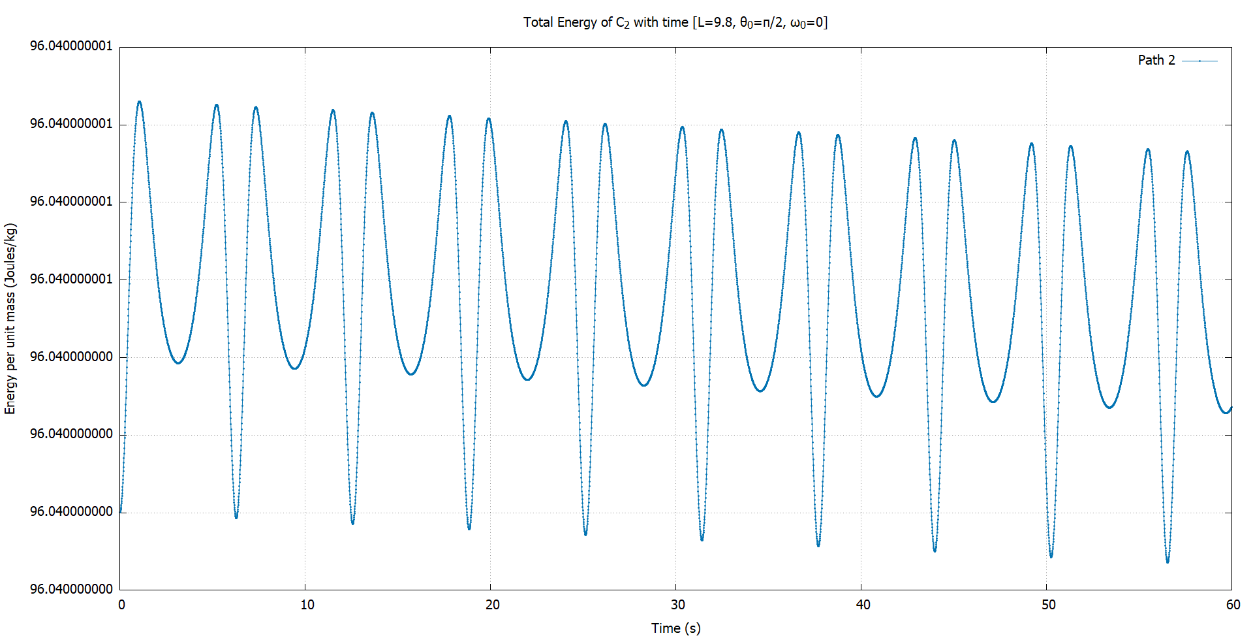
Again, no surprises here. As the driving force is increased, the system goes into overdrive, and the oscillation amplitude behaviour is somehow the reverse of the previous one. Note that the maximum amplitude is restricted for both paths. This is because there is a damping force that is present in all cases.

So, does all of the above arguments definitively rule out chaos in the given curves? Unfortunately, no. But as far as we know, these systems don’t exhibit chaos, at least via the period doubling route. To completely rule out chaotic motion, we need more rigorous methods[[7]](#footnote-7) instead of looking at a small set of parameters.

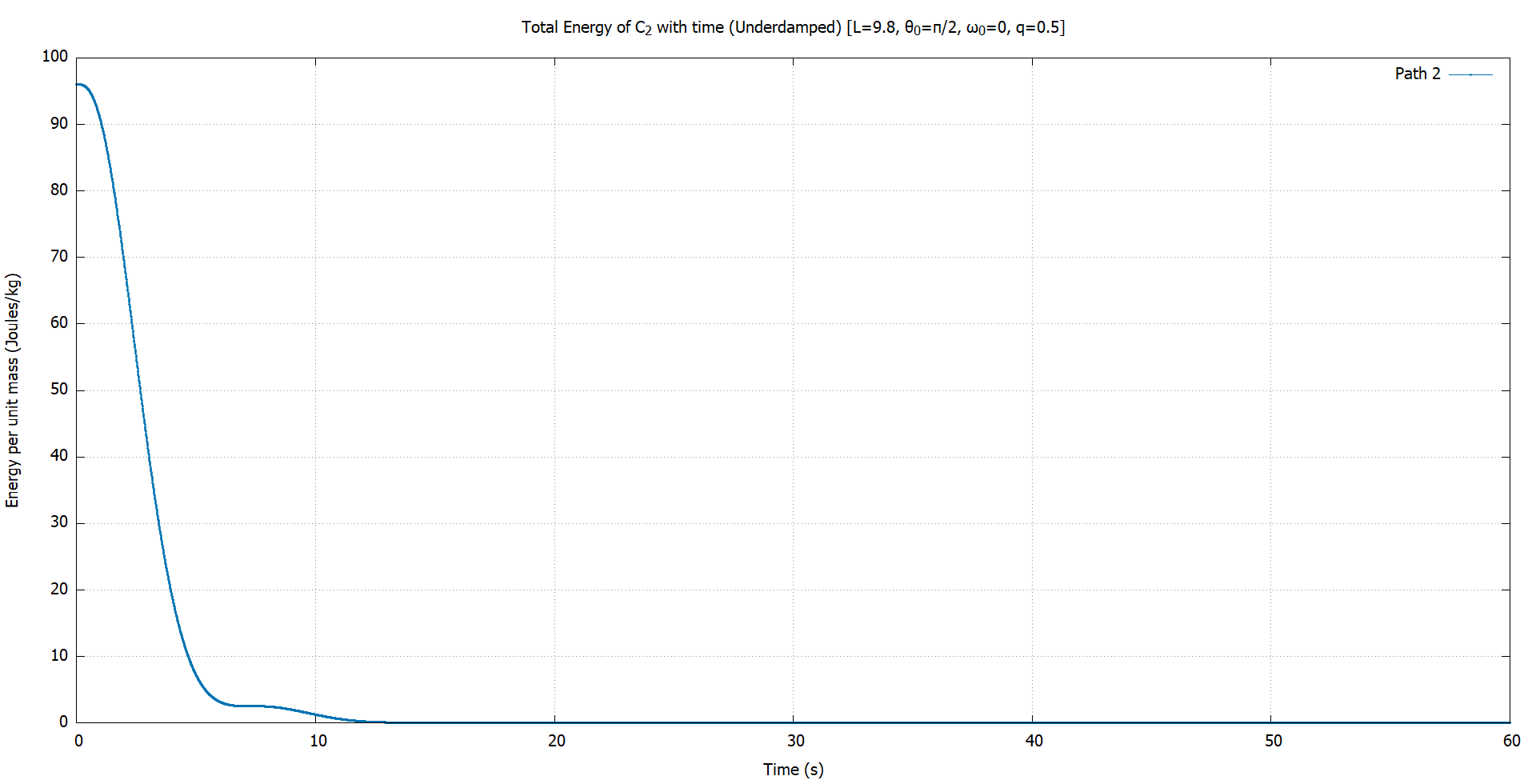
**3.7- Conservation of Energy**

We have already encountered an expression closely related to the total energy of the system in section 3.5 for :

Now, So, we have:

Below are 2 plots of total energy with time for . The first plot is un-damped:

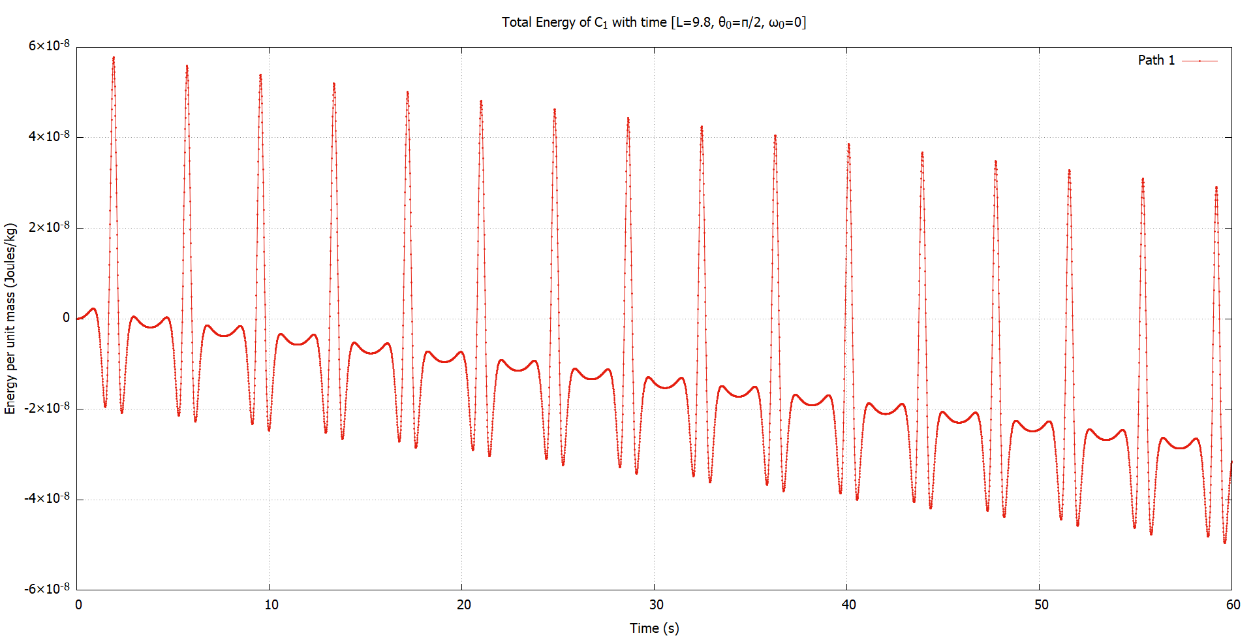
Note how little the energy variation is. The energy varies very little, and it is for all practical purposes, constant.



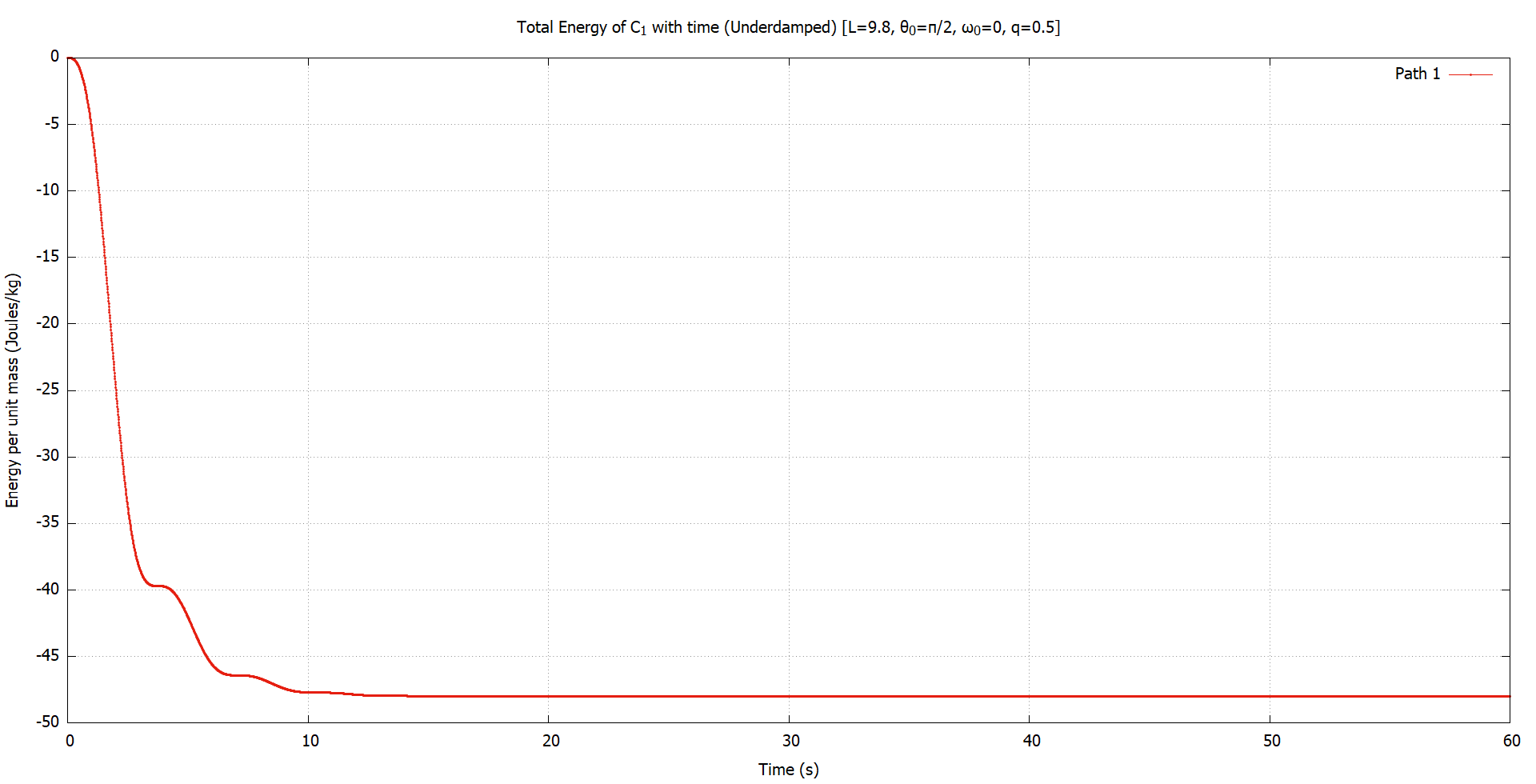
When we add a slight damping force of strength (underdamped), we get the following plot. Note how the energy decays with time and reaches 0. This corresponds to the particle being at the equilibrium point.

Similarly, for , we have:

After some algebra later, we arrive at the following expression:

And these are the corresponding plots:

Again, the variation is extremely small, and the energy is almost constant.

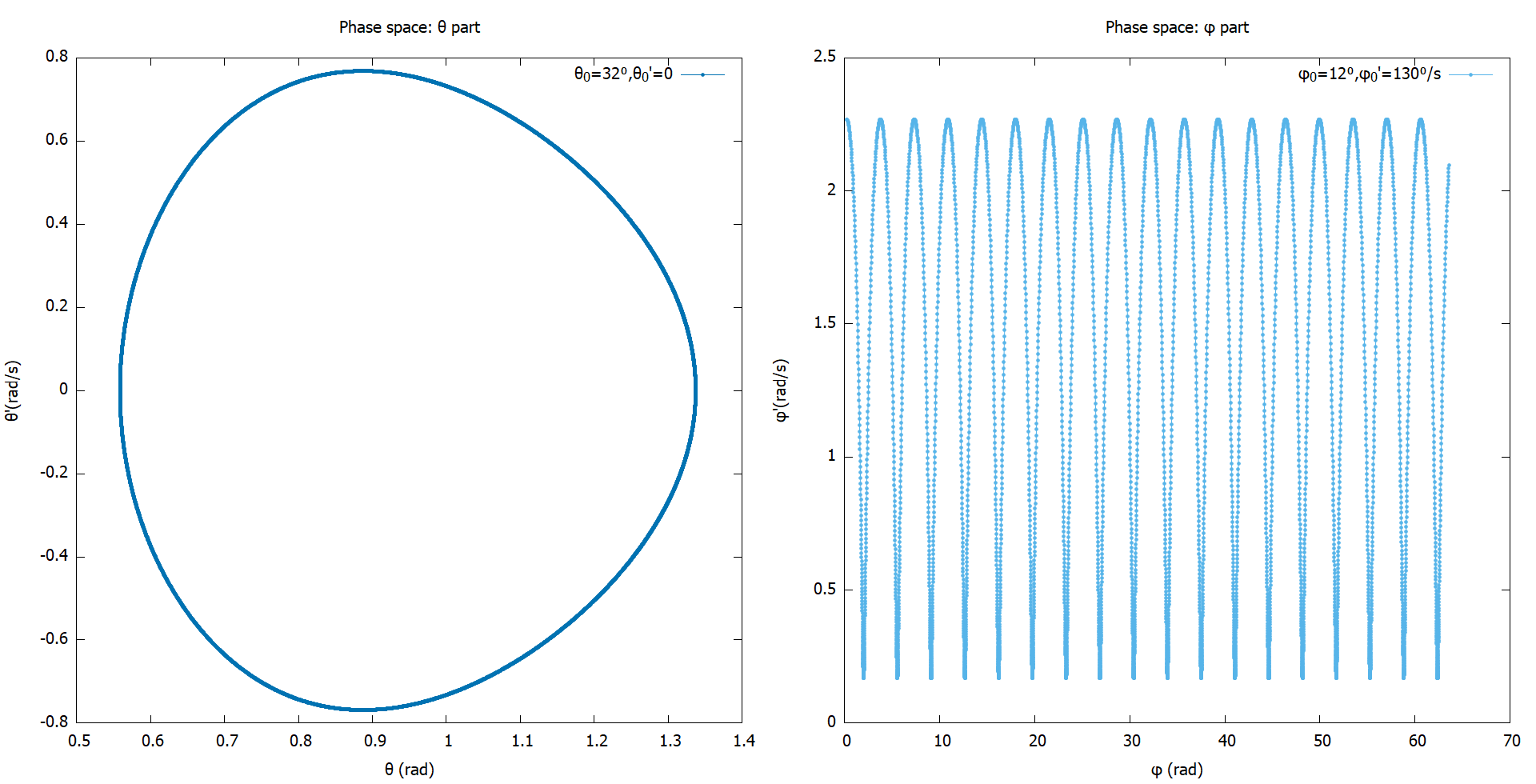
When we add damping () to the same system, we get the following plot:

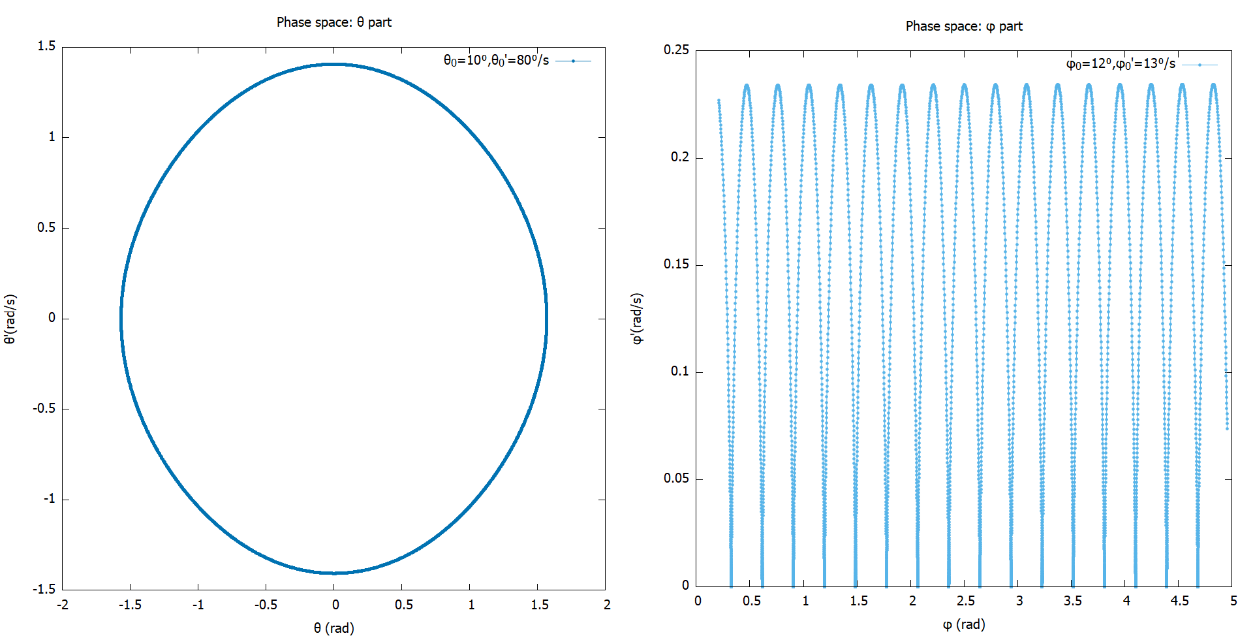
But this time, the energy doesn’t settle at 0 but at some negative value. But fear not, as this is just the gravitational potential energy at the equilibrium point, which is at () for .

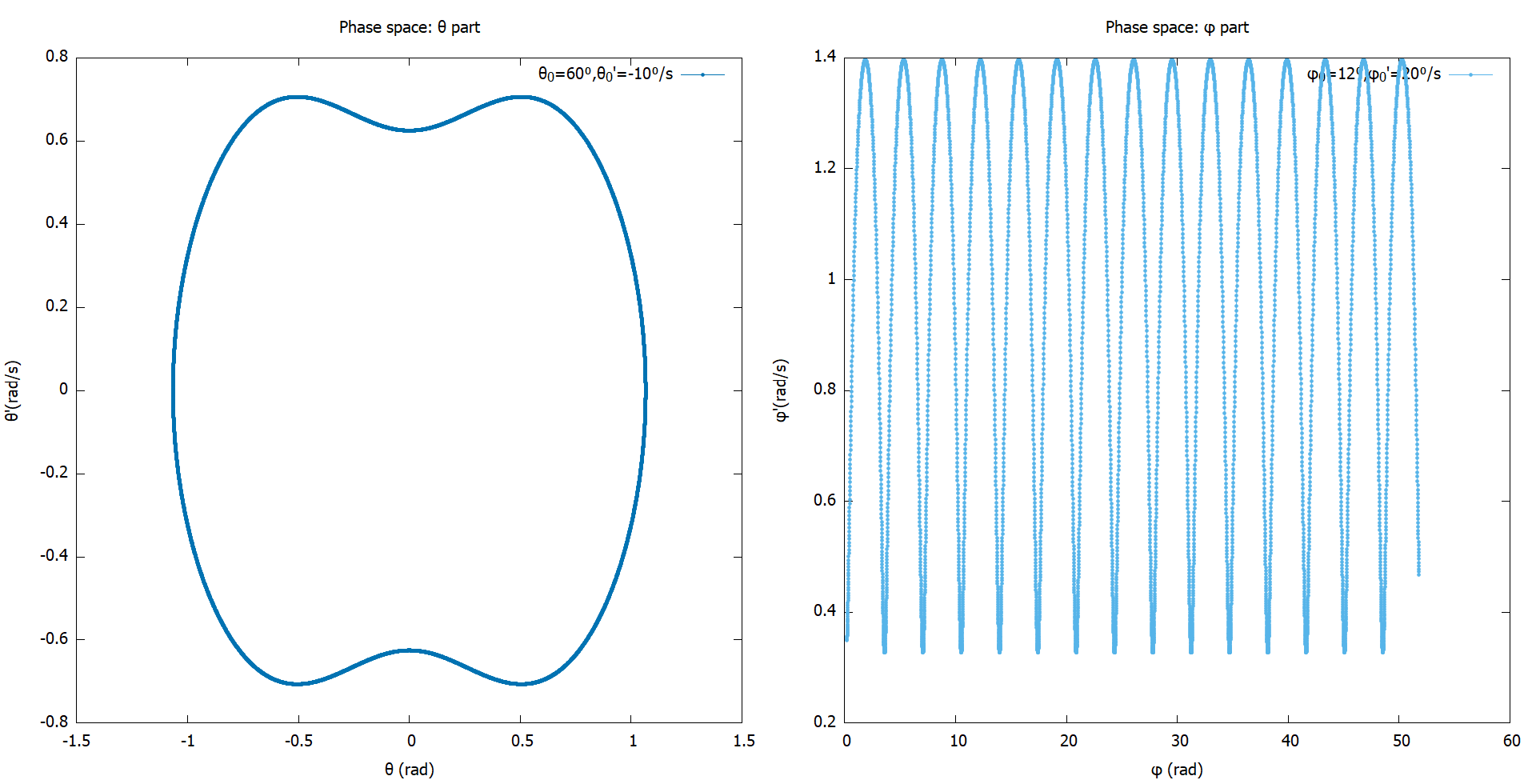
Overall, we can say that the total energy for both curves have the expected behaviour when we simulate them with RK4 method. So, the choice of RK4 was judicious.

1. **Part II: Constrained motion in 2D**

**4.1- Motion on the sphere**

 We started looking at motion on the sphere by trying out different initial conditions and looking at the corresponding phase space plots. One thing we found early on was that was not bound between , as stated in the project document. It was actually in the range . We also found quite a few interesting behaviours that would not be physical in real life, but do occur in simulation. But first, let’s have a look at some phase space plots.





Note that this is a 3D system with a constraint. So, we have 2 degrees of freedom in total, giving us a 4D phase space. But representing a 4D figure on a 2D paper is not practical. So, we resorted to splitting the phase space into 2 branches (one for , another for ). It is clear that motion along is oscillatory, and almost always periodic, as evidenced by the closed path in phase space. The motion in is more interesting. increases constantly, whereas the angular velocity oscillates between a certain range.

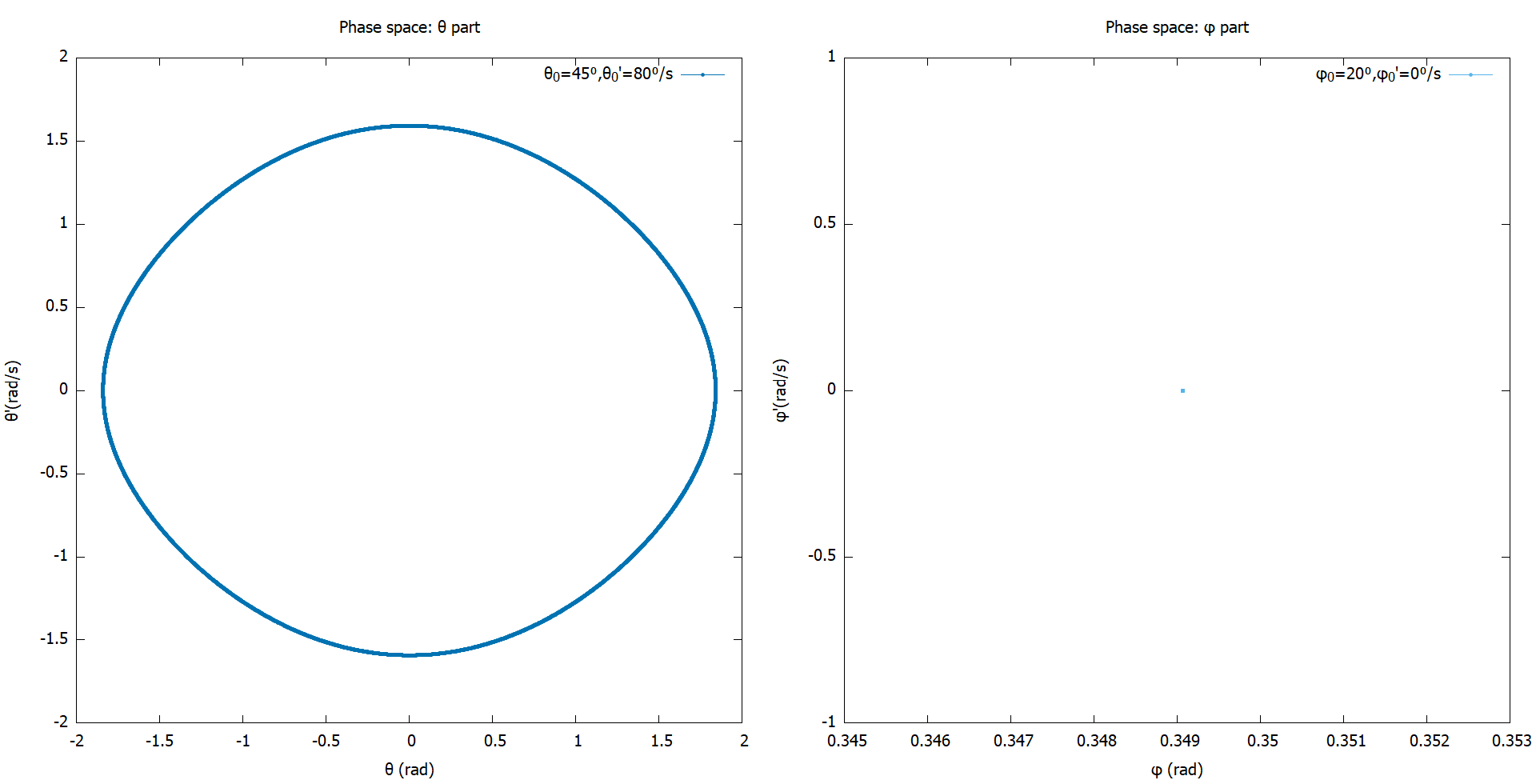
We must note that giving the system too much initial velocity results in infinities (something we encountered in in part I). But the interpretation of this is not entirely clear. We suspect that this could be an artefact of simulation or just our setup of the equations.

Now, back to the unphysical behaviour we mentioned earlier. If we place the particle at the top of the sphere (), and setting the initial angular velocities to 0 (), the particle stays put at that point. This is contrary to what would occur in real life. We would expect that the particle would drop off the top due to gravity and bounce around. This is again due to the way we set up our system.

This is also seen when we set . Even here, we would expect the particle to fall off if no velocity is given initially. But instead, we get pendulum-like oscillations.

Maybe, interpreting these equations as motion along a spherical surface may not be accurate. A more fitting description would be to describe a mass stuck to a rigid rod of length pivoted about the origin.

**4.2- Circular path: Recreating the simple pendulum motion**

 As we hinted in the previous section, we can indeed recreate the motion of a simple pendulum in a spherical surface. To do so is quite simple. Just set the angular velocity along to be 0 (). The rest of the initial conditions can be freely chosen[[8]](#footnote-8). We can confirm that the motion is indeed along a plane by plotting the phase space.

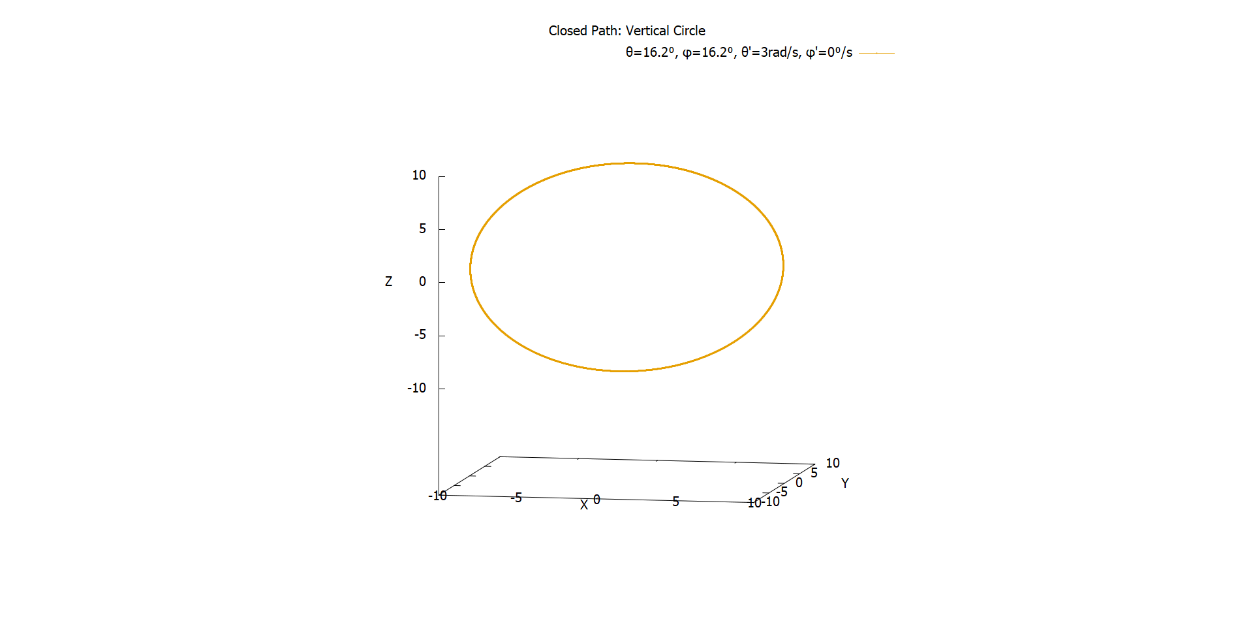
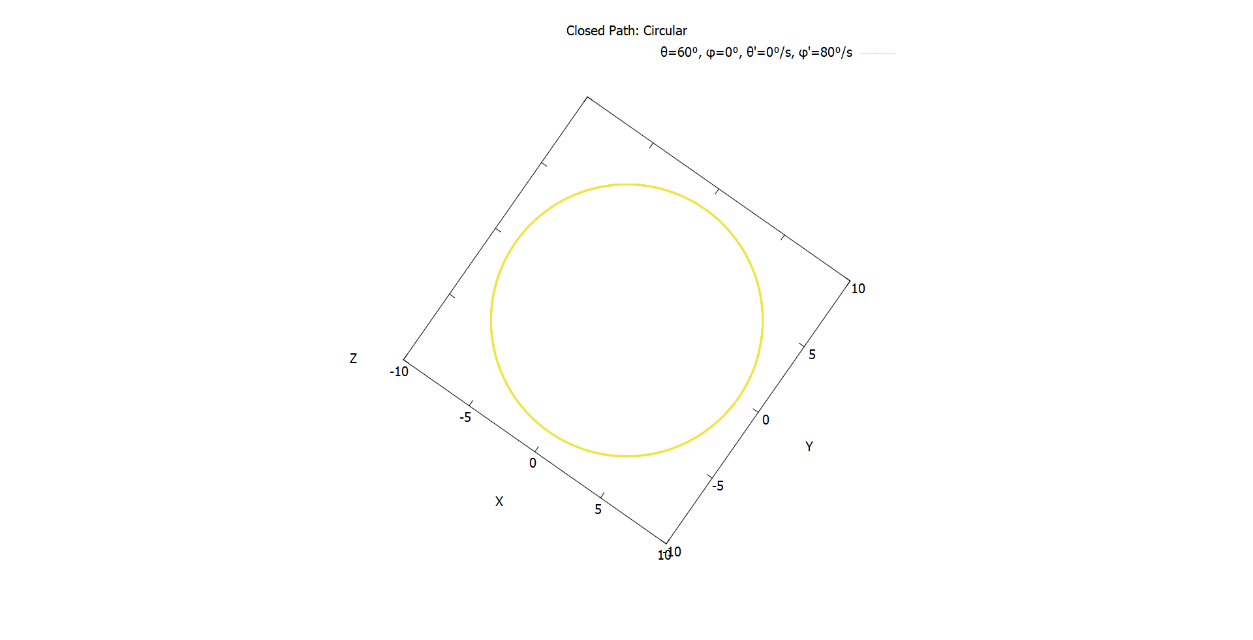
Notice that the branch of the phase space is just a single point, whereas the branch is what we typically associate with oscillatory motion. The effective degree of freedom is 1, just like the simple pendulum.

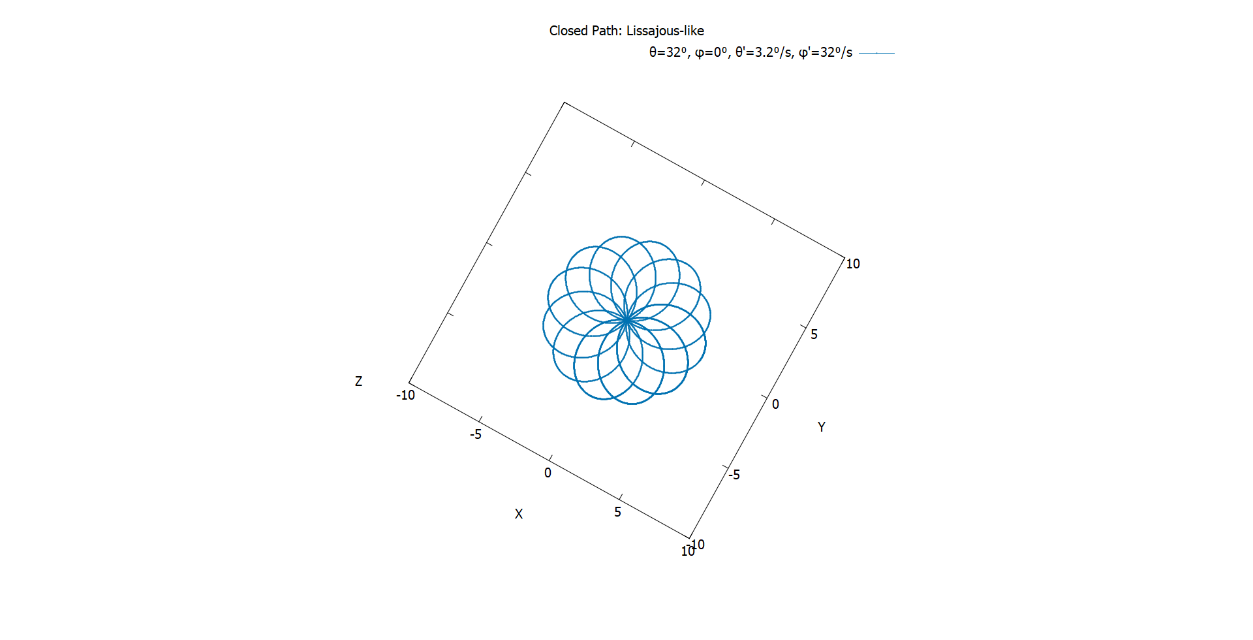
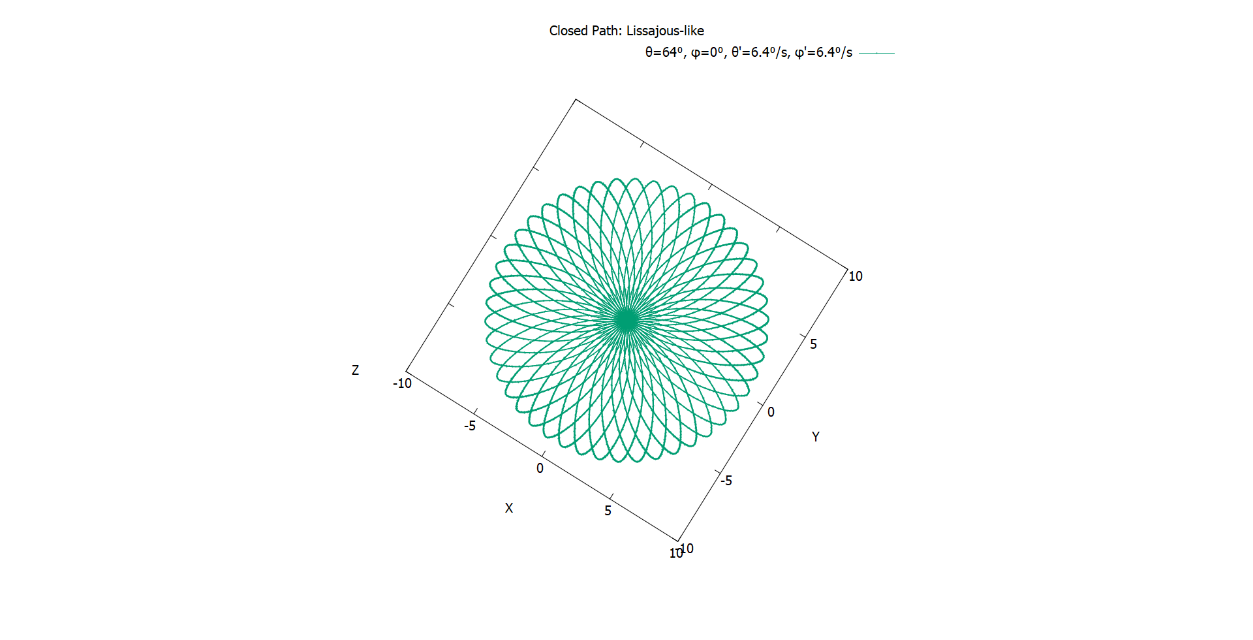
**4.3- Closed paths**

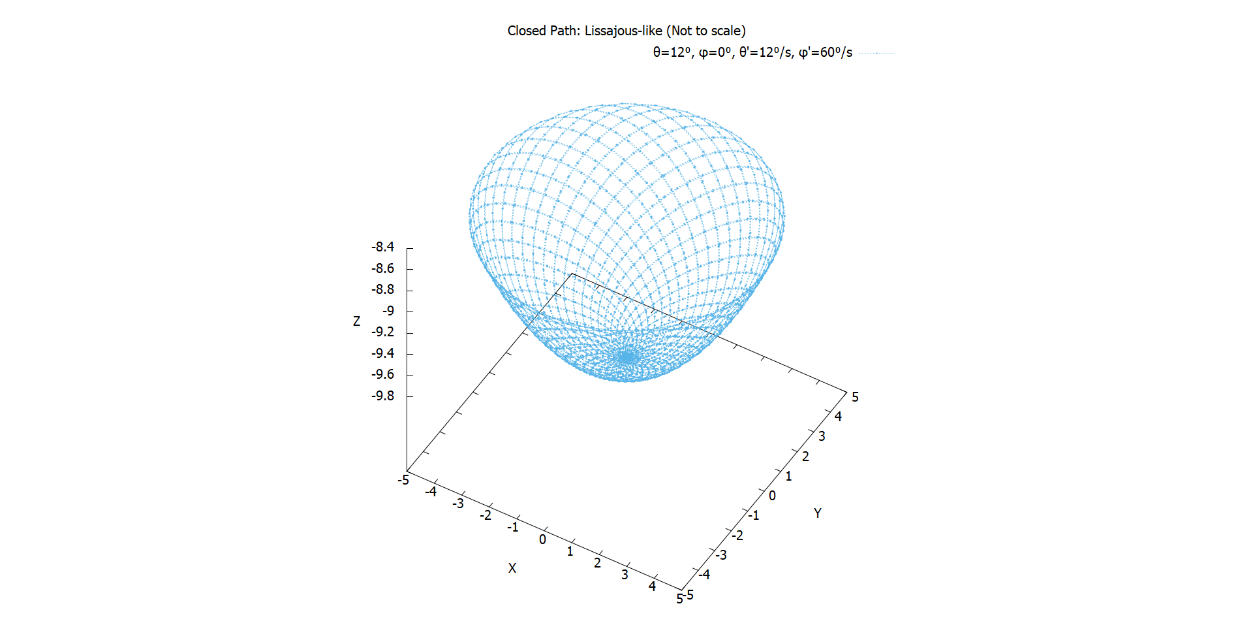
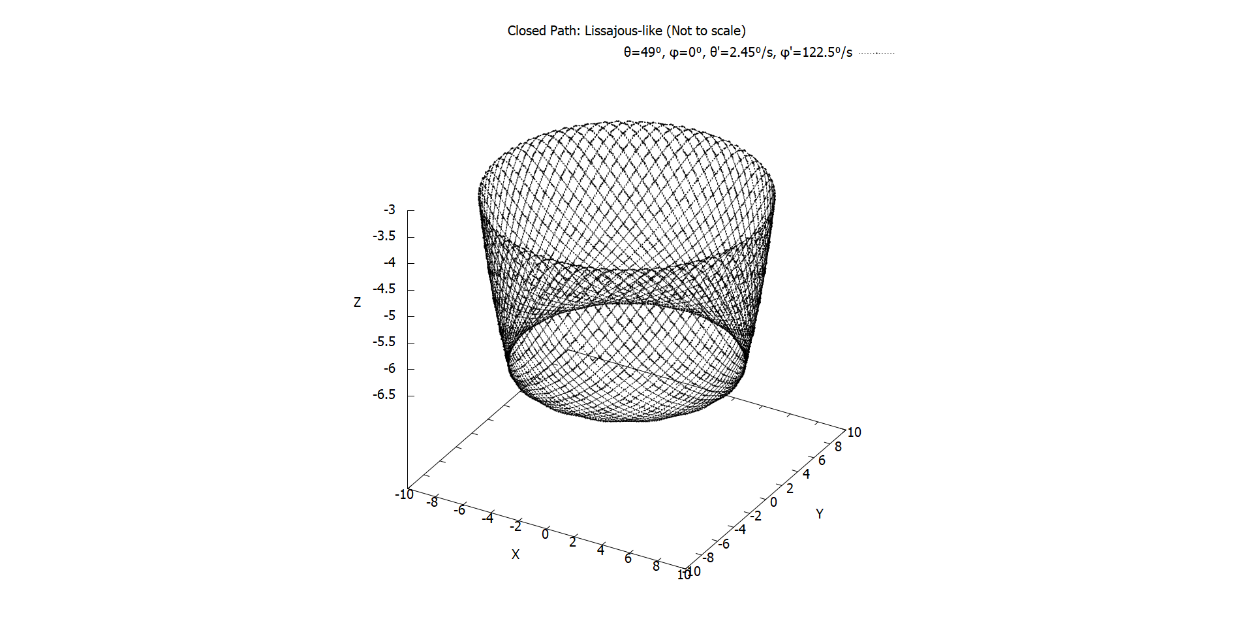
One of the obvious ways to achieve a closed path is simply to follow up from the previous section and increase the velocity to a high enough value so that the circular path becomes closed. This is exactly what we did. But we didn’t stop there. We found out that we can have closed circular path in arbitrary planes, if we adjust the velocities just right.

We even found closed paths that were not circular in our search for Lissajous-like figures. But the initial conditions for those are not at all trivial. Unlike Lissajous figures in the plane, where making the frequencies integer multiples of each other gave us a closed path, here it is not obvious what the condition is. In all cases, the paths we got were symmetric about .

How did we ascertain that the paths are indeed closed? We simulated the paths for very long times (), and observed if the long-term path is distinct and unchanging. Paths that are not closed fill up the entire region the initial energy allows them to. So, this is a sufficient test for closed paths.

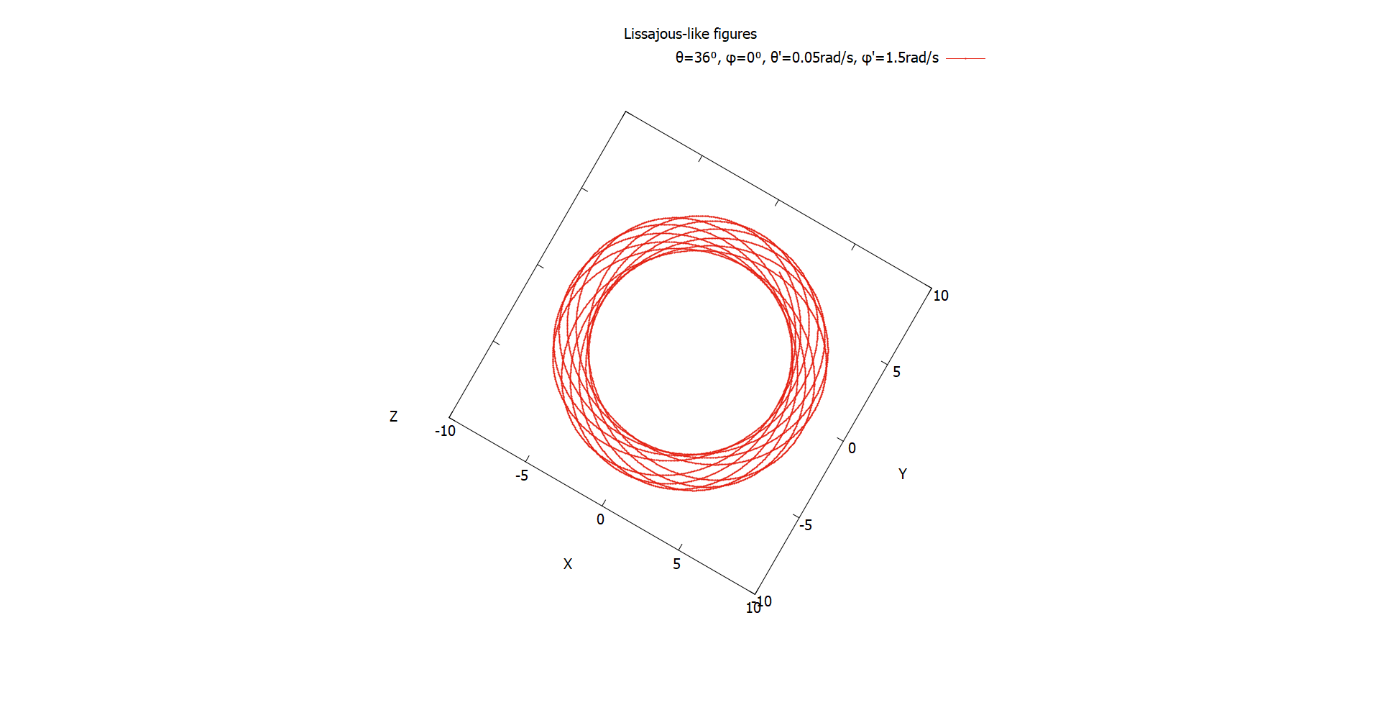
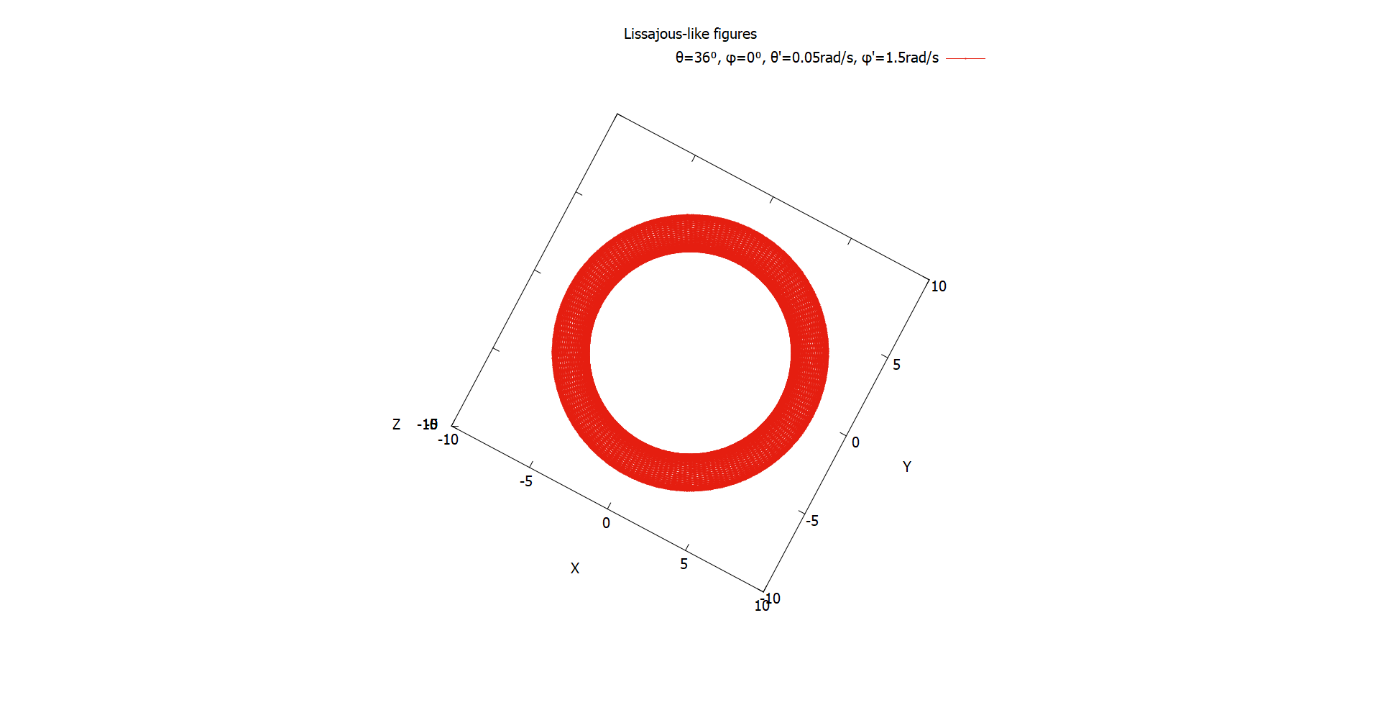
Below are some of the closed paths that we came up with, along with the initial conditions that lead to them[[9]](#footnote-9)[[10]](#footnote-10).





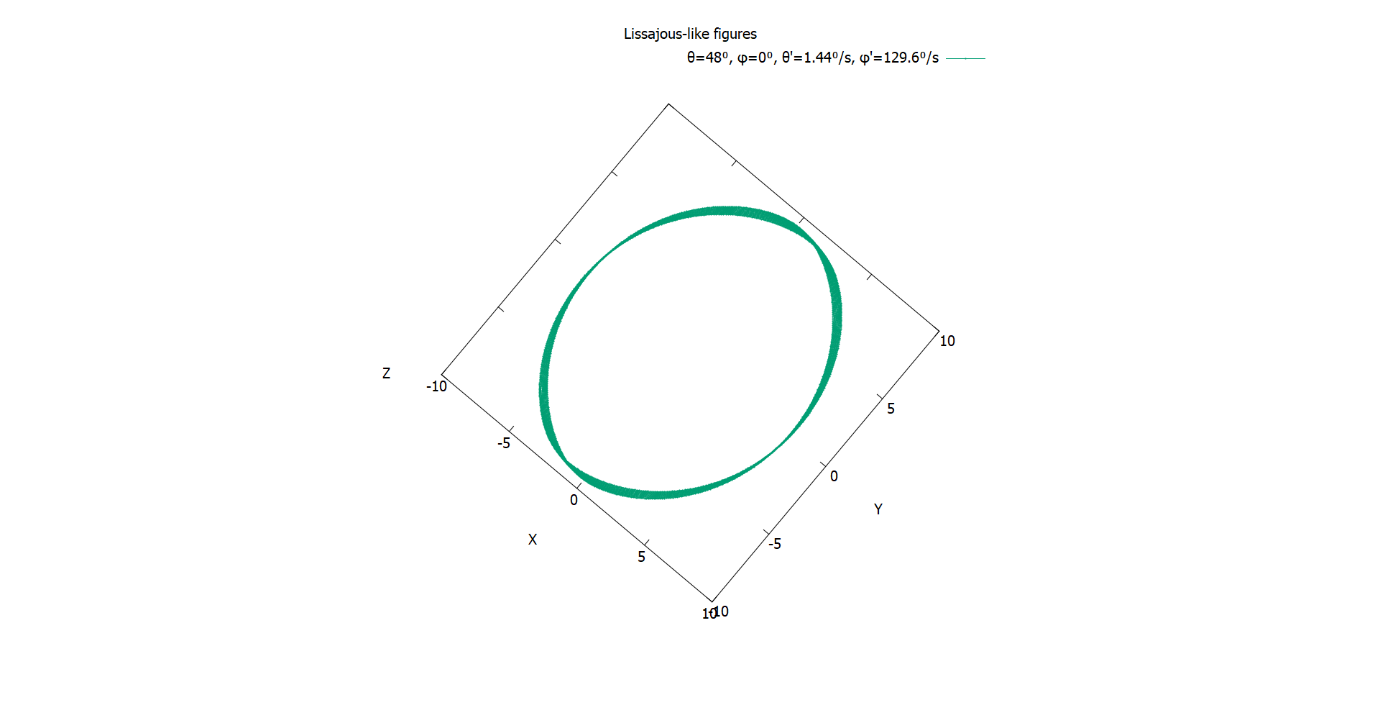
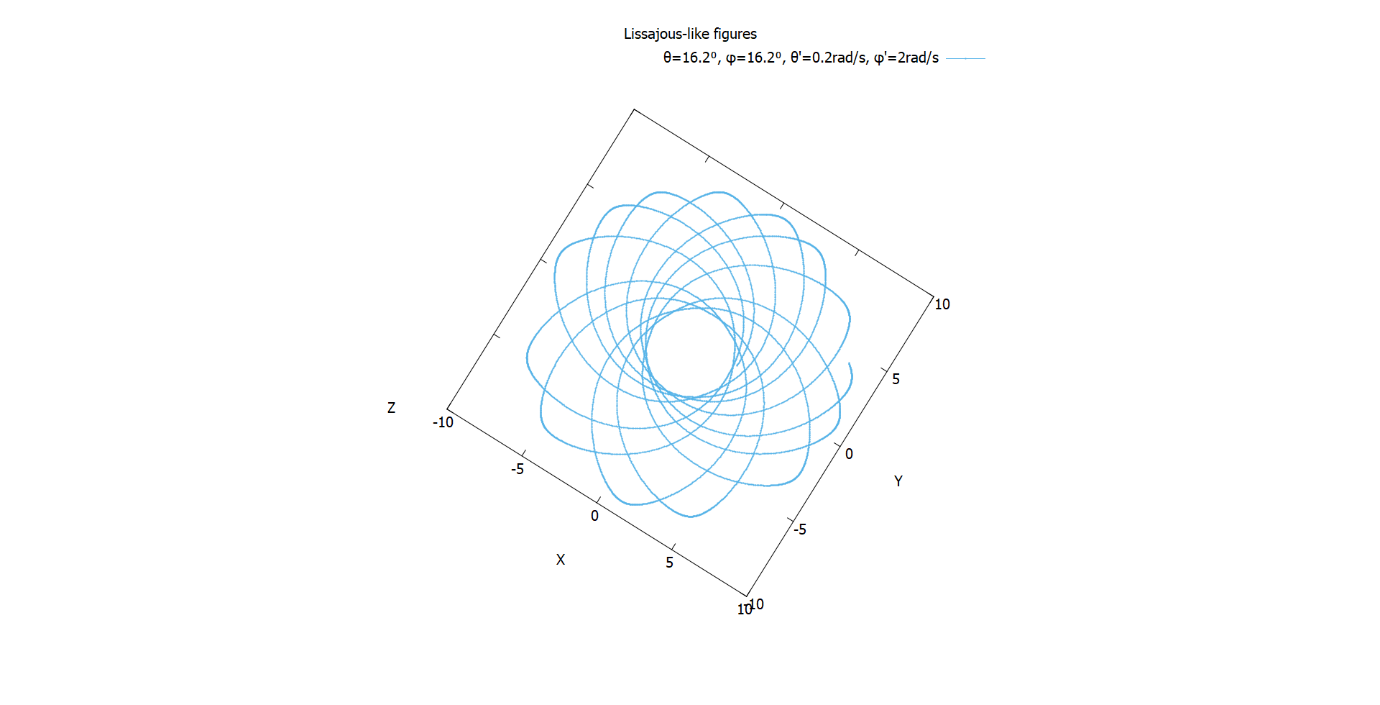
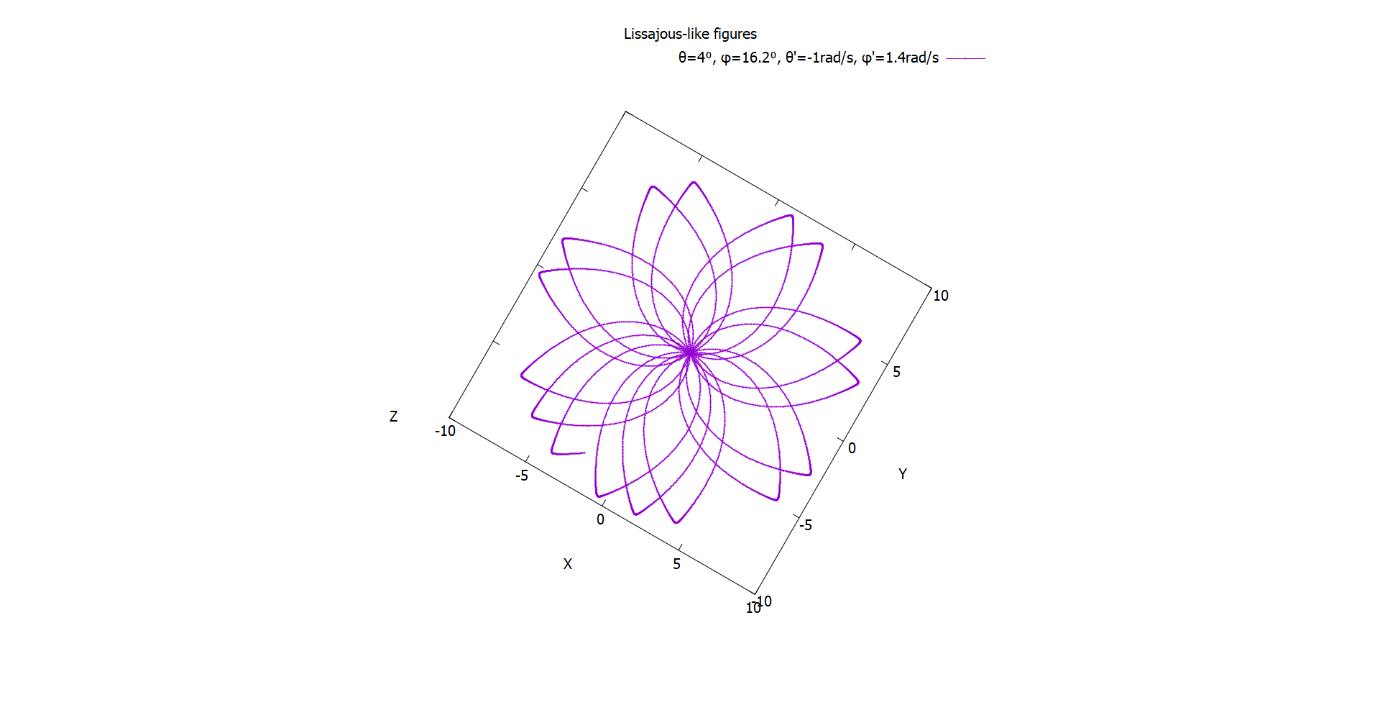
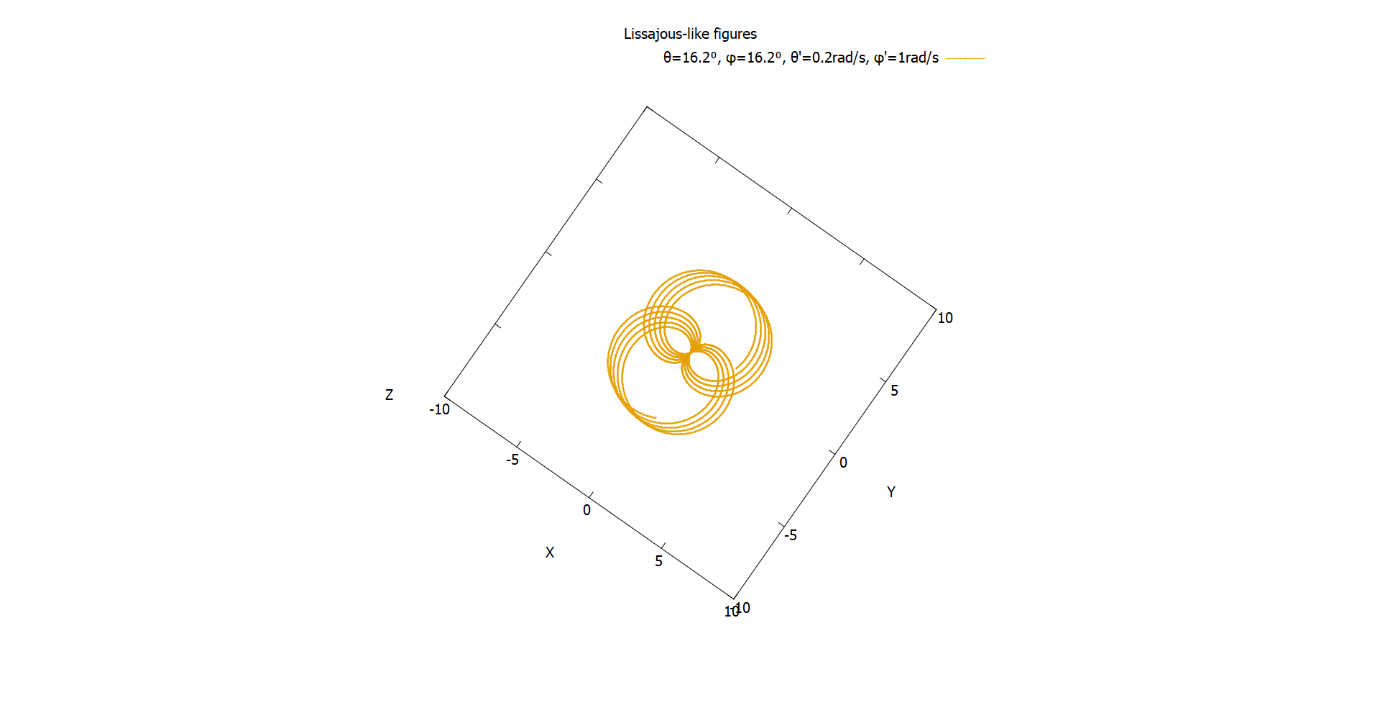
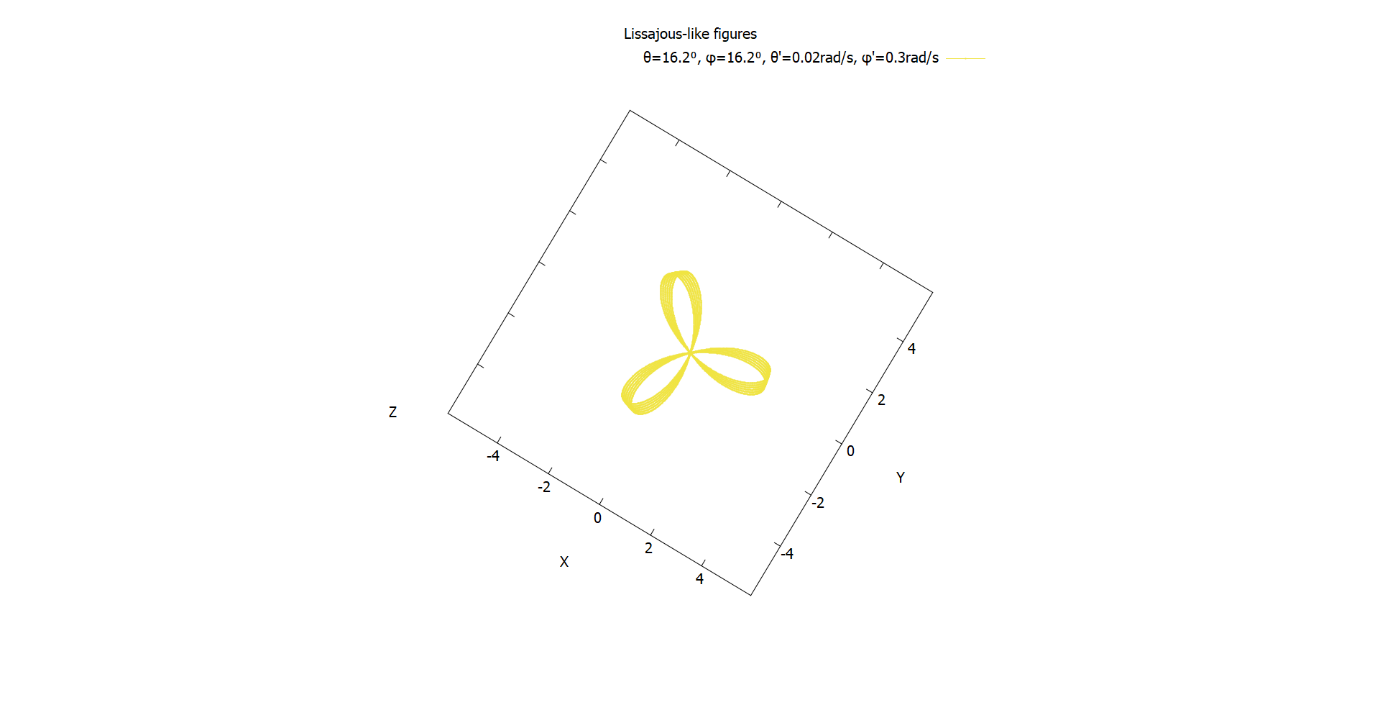
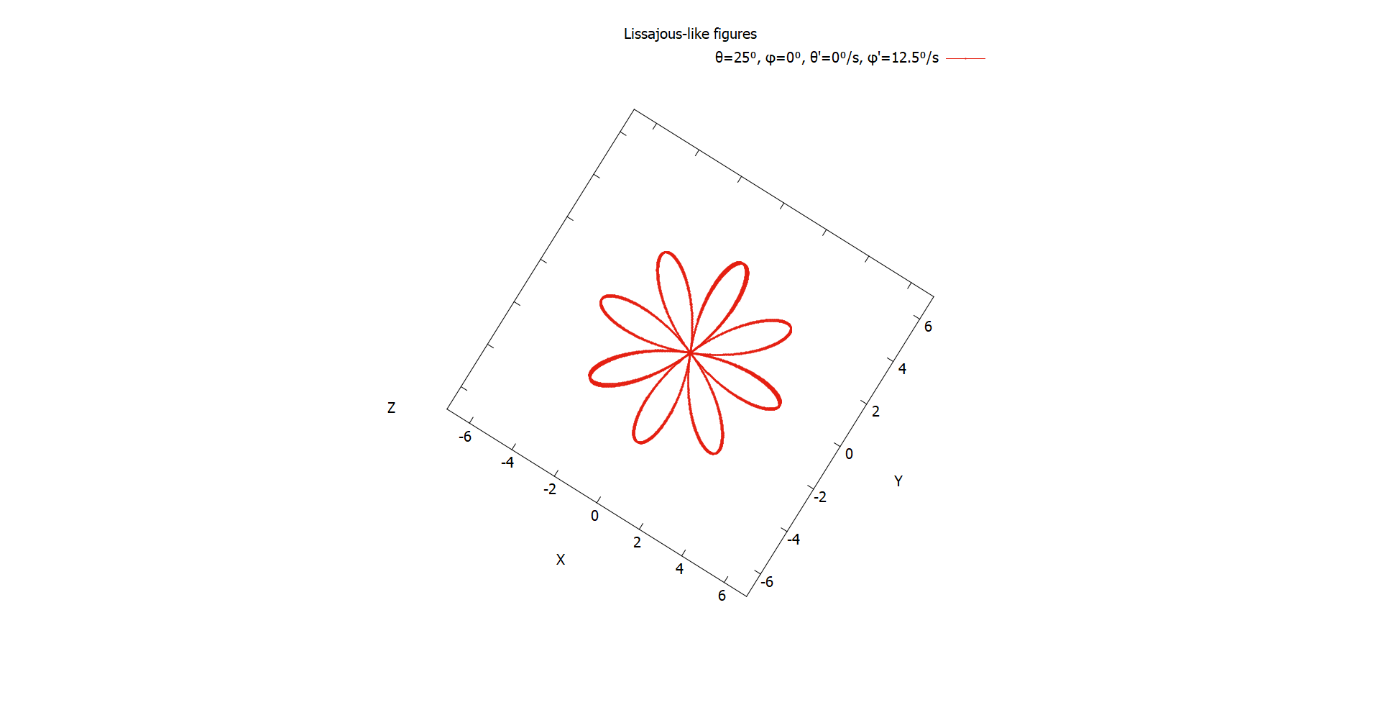
**4.4- Lissajous-like figures**

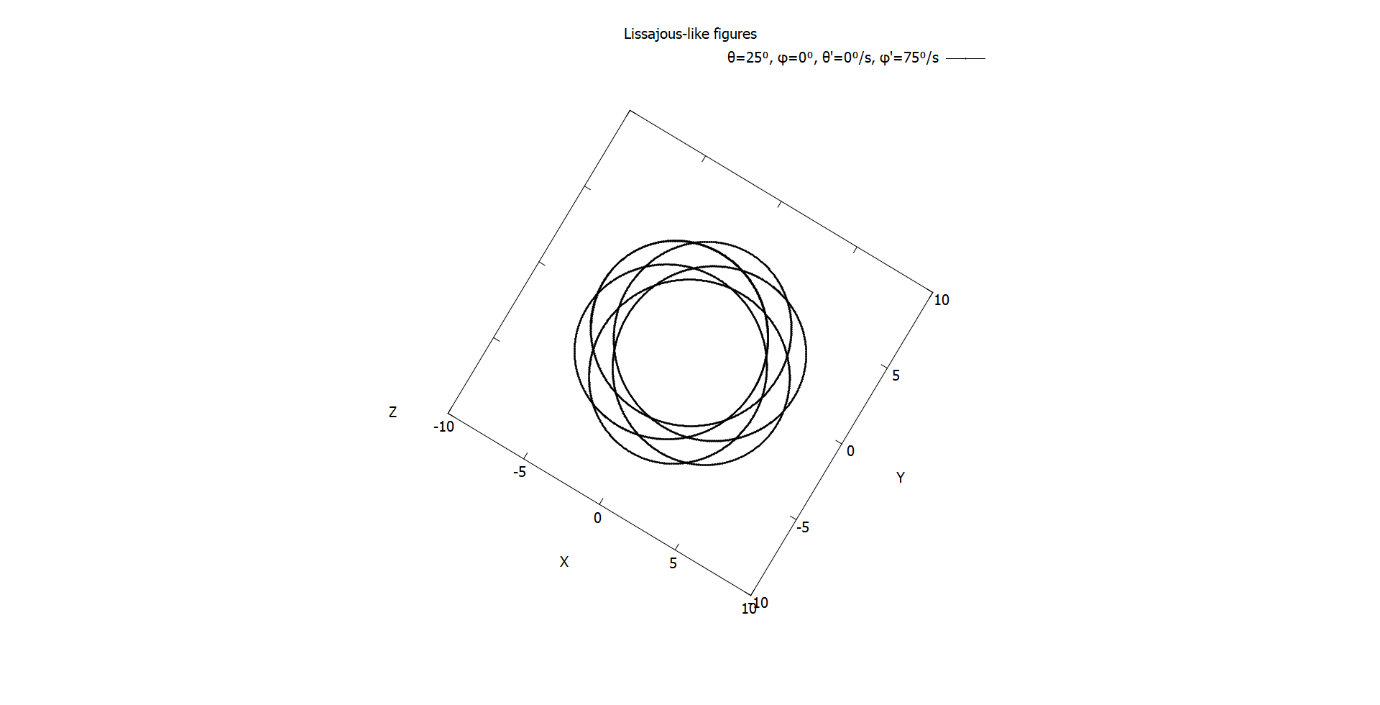
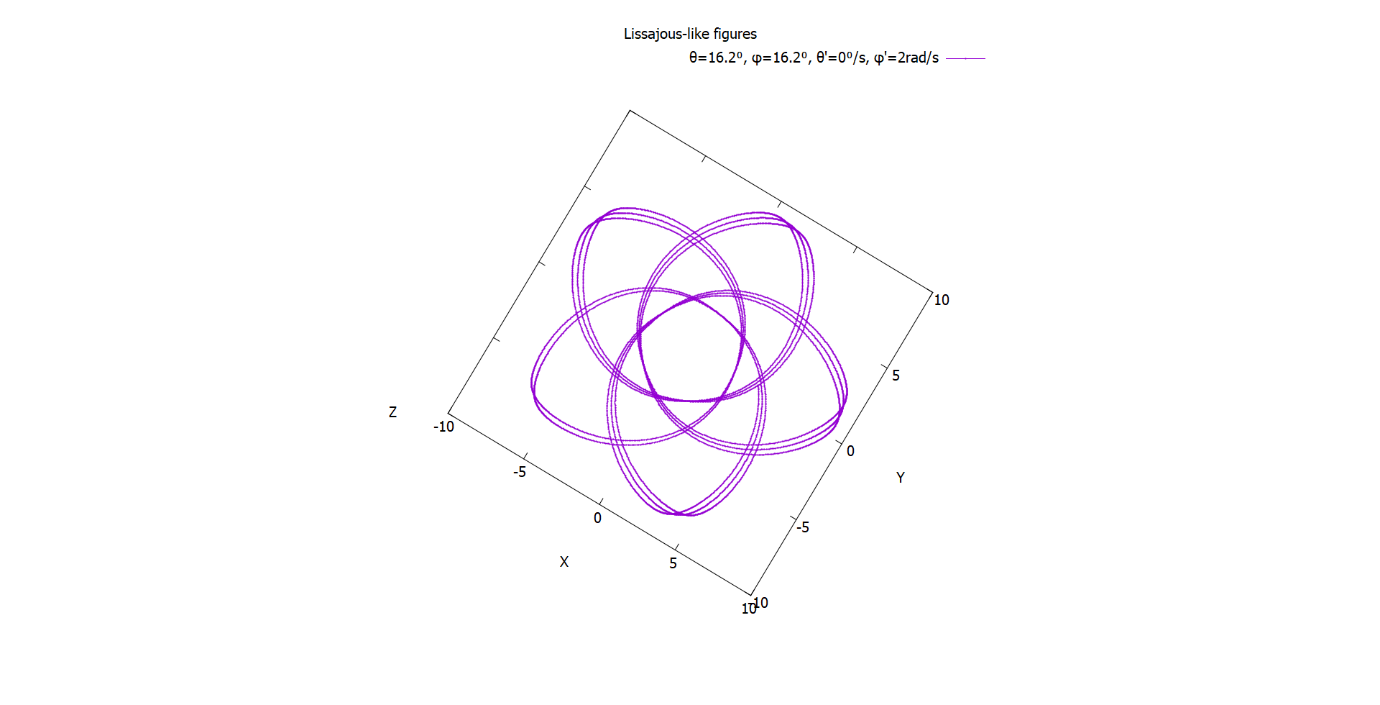
Firstly, what is really meant by “Lissajous-like” figures? If we are being honest, it is open to interpretations. Any given path can be considered “Lissajous-like”. This is compounded by the fact that there also exists spherical Lissajous figures, that have applications in MRIs. Whether those can be reproduced in our system is unclear.

 So, to be more specific, we took inspiration from 2D Lissajous figures to come up with our very own “Lissajous-like” figures. All of them have a periodic structure about . And most of them are not closed (we already saw some of the closed ones). If we allow the simulation to run for prolonged durations, we see that the path covers a section of the spherical surface entirely. This behaviour is observed in the planar Lissajous figures when the ratio of frequencies is irrational. The paths below illustrate this point. We ran the simulation for the same initial condition for 60 seconds and 2000 seconds respectively.

The patterns we got have quite intricate structures- some resemble flowers with numerous petals, whereas others look like baskets (like the plot above) that have a hole in the centre. But unfortunately, we cannot predict the shape just from the initial conditions.

In the following page, we showcase some of the “Lissajous-like” figures that we obtained along with the initial conditions that gave rise to them. Note that in all cases, the radius of the sphere was , and the simulation was run for seconds.





**4.5- (Where is) Chaos**

When looking for chaotic motion in physical systems, one of the first thing we look for is aperiodicity. That, followed by sensitivity to initial conditions are markers of a system that is chaotic. But so far, with all the initial conditions that we went through looking for closed paths and “Lissajous-like” figures, we haven’t encountered either of these. Again, this does not decisively tell us that our system is not chaotic.

But we have further reasons to believe that this system isn’t chaotic. For a simple pendulum, chaotic behaviour arises only when damping and driving forces are introduced. Here we have neither. By restricting our motion to the axis, we essentially get the simple pendulum motion back. In the absence of driving or damping forces, there is no reason to think the system is chaotic.

To be clear, we are not ruling out chaos entirely as we weren’t able to investigate it rigorously. But from all the initial conditions we tried, there doesn’t appear to be any markers of chaos.

**4.6- Additional Tasks**

1. Assuming the roller coaster was well maintained [↑](#footnote-ref-1)
2. Unless the car is in a rally race [↑](#footnote-ref-2)
3. Unless mentioned otherwise, angles are in rad, angular velocities are in rad/s, time is in seconds [↑](#footnote-ref-3)
4. See: [Is the tautochrone curve unique?: American Journal of Physics: Vol 84, No 12 (scitation.org)](https://aapt.scitation.org/doi/10.1119/1.4963770) [↑](#footnote-ref-4)
5. The values of critical damping coefficients are not exact [↑](#footnote-ref-5)
6. In all the amplitude plots, the amplitude was measured after a set time to account for the transient. [↑](#footnote-ref-6)
7. More on this in the discussion [↑](#footnote-ref-7)
8. Given the velocity is not too high [↑](#footnote-ref-8)
9. refer to the angular velocities in the figure [↑](#footnote-ref-9)
10. Some of the figures have been rescaled for clarity [↑](#footnote-ref-10)